ASPECTS OF FOUR DIMENSIONAL $\mathcal{N} = 2$ FIELD THEORY

A Dissertation

by

DAN XIE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2011

Major Subject: Physics

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Approved by:

Chair of Committee,	Dimitri V. Nanopoulos
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ABSTRACT

Aspects of Four Dimensional $\mathcal{N} = 2$ Field Theory. (August 2011)

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Chair of Advisory Committee: Dr. Dimitri V. Nanopoulos

New four dimensional $\mathcal{N} = 2$ field theories can be engineered from compactifying six dimensional $(2, 0)$ superconformal field theory on a punctured Riemann surface. Hitchin's equation is defined on this Riemann surface, and the fields are singular at the punctures. Four dimensional theory is entirely determined by the data at the punctures and theory without lagrangian description can also be constructed.

We first constructed new four dimensional generalized superconformal quiver gauge theory by putting regular singularity at the puncture. The algorithm of calculating weakly coupled gauge group in any duality frame is developed. The asymptotical free theory and Argyres-Douglas theory can also be constructed using six dimensional method which requires introducing irregular singularity.

Compactifying previous four dimensional theory down to three dimensions, the corresponding $\mathcal{N} = 4$ theory has the interesting mirror symmetry property. The mirror theory for the generalized superconformal quiver gauge theory can be derived using the data at the puncture, too. Motivated by this construction, we studied the other three dimensional theories deformed from above theory and found their mirrors.

The surprising relation of above four dimensional gauge theory and two dimensional conformal field theory may have some deep implications. The S-duality of four dimensional theory and the crossing symmetry and modular invariance of two dimensional theory are naturally related.

To my family

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CHAPTER I

INTRODUCTION

The method of solving four dimensional $\mathcal{N} = 2$ SU(2) gauge theories was proposed by Seiberg and Witten [1, 2] more than fifteen years ago. The exact solution provides tremendous amount of non-perturbative information, which is extremely valuable in studying dynamics of quantum field theory. A lot of conjectures about the quantum dynamics of field theory are confirmed for the first time. It also provides new topological invariant of four manifolds [3]. It has interesting relations with other branches of physics, e.g. the classical integrable system [4, 5, 6].

Since then, the solutions for many other $\mathcal{N} = 2$ quantum field theories have been found [7, 8, 9]. It is surprising that a major breakthrough about $\mathcal{N} = 2$ theories has occurred in last couple years after so many years' intense study. The breakthrough begins with the understanding of S duality property of certain superconformal field theory (SCFT) [10, 11], this opens the door of studying non-lagrangian theories which are a crucial ingredient for the duality. Then the method of determining the exact stable BPS spectrum and its wall crossing behavior is discovered [12, 13, 14]. Another line of development deals with the relation of gauge theory with the quantization of integrable system: it is found that the Nekrasov partition function [15] of four dimensional theory is related to the quantization of the corresponding integrable system associated with the gauge theory [16, 17, 18]; The Alday-Gaiotto-Tachikawa (AGT) conjecture [19] is another quantum relation which relates the Nekrasov partition function with the conformal block of two dimensional conformal field theory (CFT).

This dissertation follows the style of *Journal of High Energy Physics*.

The origin of all these exciting development is the mysterious six dimensional $(2,0)$ SCFT [11]. This theory is strongly coupled and has no lagrangian description, little is known about this theory [20]. However, four dimensional supersymmetric gauge theory can be engineered by compactifying six dimensional theory on a Riemann surface. If the Riemann surface is a torus, then the four dimensional theory is the famous $\mathcal{N}=4$ super Yang-Mills theory. The gauge coupling is simply the complex structure of the torus, since the six dimensional theory is conformal, the four dimensional theory only depends on the complex structure moduli of the torus, therefore, the modular group of the torus is just the S-duality group of the gauge theory!

Given the beautiful interpretation of the S-duality of four dimensional gauge theory in terms of the geometric property of the Riemann surface, it is natural to extend the same analysis to the four dimensional $\mathcal{N}=2$ SCFT. There must be some new structures defined on the Riemann surface: first marked points are introduced; second the Hitchin's equation is defined on the Riemann surface and the solution of the Hitchin's equation is singular at these punctures. Then the S-duality of four dimensional $\mathcal{N}=2$ theory is realized as the modular group of the punctured Riemann surface, moreover, the Seiberg-Witten curve can be constructed from the data at the puncture. The Hitchin's moduli space is an integrable system in one of complex structure, so an integrable system is naturally associated with a four dimensional field theory. This also provides the motivation for AGT conjecture. The Liouville theory is defined on the punctured Riemann surface and the AGT conjecture is naturally studied in this context.

It is then interesting to study in detail this six dimensional construction. New four dimensional $\mathcal{N}=2$ field theory can be engineered from compactifying six dimensional $(2,0)$ superconformal field theory on a punctured Riemann surface. Hitchin's equation is defined on this Riemann surface and the fields in Hitchin's equation are singular at the punctures. Four dimensional theory is entirely determined by the data at the puncture. It is remarkable that theory without lagrangian description can also be constructed in this way, which leads to a lot of surprising result.

We first construct new four dimensional generalized superconformal quiver gauge theory by putting regular singularity at the puncture. The Seiberg-Witten curve, S-duality property of the theory can be easily derived from the geometric property of the Riemann surface. The algorithm of calculating weakly coupled gauge group in any duality frame is developed.

The asymptotical free theory and Argyres-Douglas field theory can also be constructed using six dimensional method. This requires introducing irregular singularity of Hithcin's equation. The specific form of irregular solution is worked out for a large class of asymptotical free theories.

Compactify four dimensional theory down to three dimension, the corresponding $\mathcal{N}=4$ theory has the interesting mirror symmetry behavior. The mirror theory for the generalized superconformal quiver gauge theory can be derived using the data at the puncture too. Motivated by this construction, we study other three dimensional theories deformed from the above theory and find their mirrors.

The surprising relation of four dimensional gauge theory and two dimensional conformal field theory may have some deep implications. The S-duality of four dimensional theory and the crossing symmetry and modular invariance of two dimensional theory are naturally related.

This dissertation is organized in the following way. In Chapter II, a brief review of Seiberg-Witten theory is given, the integrable system approach and the string theory construction is emphasized. In Chapter III, the six dimensional construction is introduced. In Chapter IV, detailed study of regular singular solution of Hitchin's equation is given; the generalized superconformal quiver gauge theory is discussed and the gauge group in any weakly coupled frame is worked out. Chapter V studies the irregular singular solutions to Hithcin's equation and uses them to describe asymptotical free theory and Argyres-Douglas theory. In Chapter VI, the three dimensional mirror symmetry of the generalized quiver superconformal gauge theory and its extension is discussed. Chapter VII deals with the relation between four dimensional gauge theory and two dimensional conformal field theory. We conclude with some discussion and open problems in Chapter VIII.

Part of the material presented in Chapter III is based on [21]. Chapter IV is mainly based on the author's paper [22, 23]. Chapter V is taken from [24] and Chapter VI is from those two papers [25, 26]. Chapter VII is taken from the paper [27].

CHAPTER II

SEIBERG-WITTEN THEORY

In this chapter, we will review the general aspects of four dimensional $\mathcal{N}=2$ supersymmetric field theory. The superalgebra is introduced and its representation is worked out, the lagrangian for general $\mathcal{N}=2$ theory is constructed using $\mathcal{N}=1$ superspace; we also discuss how to derive the four dimensional theory from dimensional reduction of a six dimensional theory. The Seiberg-Witten solution is reviewed and its relation to integrable system and three dimensional theory is briefly discussed. String theory construction of Seiberg-Witten solution is introduced in detail.

A. Generality of four dimensional $\mathcal{N}=2$ theory

The four dimensional $\mathcal{N}=1$ supersymmetry algebra consists of the following anti-commutative relation (this section is based on [28, 29].

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu, \\ \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 0. \end{aligned} \tag{2.1}$$

Here Q_β is a Weyl spinor and $\bar{Q}_{\dot{\beta}} = Q^+_\beta$. There is a $U(1)_R$ symmetry which rotates the supercharge by a phase, this symmetry may or may not be a symmetry in the quantum theory.

Look at massive representation, we can work on the rest frame with $P^\mu = (-M, 0, 0, 0)$, then the superalgebra simplifies as

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2M\delta_{\alpha\dot{\beta}}. \tag{2.2}$$

Define normalized generator $b_\alpha = \frac{1}{\sqrt{2M}} Q_\alpha$ and $b^+_\alpha = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha}}$, the superalgebra becomes

$$\{b_\alpha, b_\beta^+\} = \delta_{\alpha,\beta},\ \{b_\alpha, b_\beta\} = \{b_\alpha^+, b_\beta^+\} = 0. \tag{2.3}$$

These are just familiar Clifford algebra. Let $|\Omega >$ be the vacua state which has the property $b_\alpha|\Omega >= 0$, the other states are $b_1^+|\Omega >, b_2^+|\Omega >, b_1^+ b_2^+|\Omega >$. If $|\Omega >$ has spin j, then these other three states have spin $j+\frac{1}{2}, j, j-\frac{1}{2}$. If $j=0$, its CPT conjugate forms another set of states with $j=0$. These are represented by two Weyl spinors and one complex scalar. If $j=\frac{1}{2}$, there are four states with spin $1, \frac{1}{2}, \frac{1}{2}, 0$, its CPT conjugate has states with spin $-1, -\frac{1}{2}, -\frac{1}{2}, 0$. These are represented by two Weyl spinors and one massive gauge fields. The number of bosonic degree of freedoms are equal to the fermionic degree of freedoms which are the typical feature of supersymmetric field theories. One can also take $j=\frac{3}{2}$, etc, in these cases, the theory includes gravity. I will only consider field theory in the following.

For massless particles, take the rest frame with $P_\mu = -E, E, 0, 0$, thus the SUSY algebra becomes

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.4}$$

Define $b_\alpha = \frac{1}{2\sqrt{E}} Q_\alpha$ and $b_\alpha^+ = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot\alpha}$. This time the only nontrivial algebra is

$$\{b_1, b_1^+\} = 1. \tag{2.5}$$

This time there are only two states for the representation of SUSY algebra with spin $j, \frac{j+1}{2}$. When $j=0$, together its CPT conjugate, there are a total of four states with spin $(-\frac{1}{2}, 0, 0, \frac{1}{2})$, these are represented by a Weyl spinor and a complex scalar. This multiplet is called Chiral multiplet. One should be a little bit careful, since the vacua of the CPT representation maybe the highest states instead of lowest states.

If $j=\frac{1}{2}$, together with its CPT conjugate, there are states $(-1, -\frac{1}{2}, \frac{1}{2}, 1)$, which are represented by a massless gauge field and a Weyl spinor, this multiplet is called vector multiplet. The $\mathcal{N}=1$ multiplets are summarized as

$$\begin{aligned} \text{vector multiplet :} &\quad (-1,-\frac{1}{2})+(\frac{1}{2},1), \\ \text{chiral multiplet :} &\quad (0,\frac{1}{2})+(-\frac{1}{2},0). \end{aligned} \tag{2.6}$$

The four dimensional $\mathcal{N}=2$ SUSY algebra reads

$$\{Q^I_\alpha,\bar{Q}^J_{\dot\beta}\}=2\delta^{IJ}\sigma^\mu_{\alpha\dot\beta}P_\mu,\quad \{Q_\alpha,Q_\beta\}=2\epsilon_{\alpha\beta}Z^{IJ},\quad \{\bar{Q}^I_{\dot\alpha},\bar{Q}^J_{\dot\beta}\}=2\epsilon_{\dot\alpha\dot\beta}Z^{*IJ}. \tag{2.7}$$

where $I,J=1,2$, Z^{IJ} is called central charge and is antisymmetric in index I,J. There is a $SU(2)_R\times U(1)_R$ symmetry and the supercharge transforms as a doublet under $SU(2)_R$.

Consider first massless representation, we must set $Z=0$ from the unitary requirement. Along the similar line as $N=1$ case,(we just have more sets of creation operators and there are four states in one multiplet), the states has the helicity $j,j+\frac{1}{2},j+\frac{1}{2},j+1$. we have the following SUSY multiplet structure (together with its CPT conjugate)

$$\begin{aligned} \text{vector multiplet :} &\quad (-1,-\frac{1}{2}^2,0)+(0,\frac{1}{2}^2,1), \\ \text{hypermultiplet :} &\quad (-\frac{1}{2},0^2,\frac{1}{2})+(-\frac{1}{2},0^2,\frac{1}{2}). \end{aligned} \tag{2.8}$$

Notice that if the minimal itself is self CPT conjugate (for instance, this happens when the matter representation is real). we do not need two multiplets, this can happen for the hypermultiplet. $\mathcal{N}=2$ vector multiplet consists of a chiral multiplet and a vector multiplet in the $\mathcal{N}=1$ language, while $\mathcal{N}=2$ contains two $\mathcal{N}=1$ chiral multiplets. From the transformation of the supercharges under $SU(2)_R\times U(1)_R$, one can see that for a hypermultiplet, spin 0 particle transform as a doublet under $SU(2)_R$ and as a singlet under the $U(1)_R$, while fermions are singlet under $SU(2)_R$ have charge $1,-1$ under $U(1)_R$. For the vector multiplet, the fermions transform as doublet under $SU(2)$ and has charge 1 under $U(1)_R$, the scalar is a singlet under $SU(2)_R$ and has charge 2 under $U(1)_R$, the gauge bosons are the singlet under R symmetry.

For massive particles, take the rest frame with $P_\mu = (-M, 0, 0, 0)$, the SUSY algebra takes the simple form

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2M\delta_{\alpha,\dot\beta}\delta^{IJ},\ \{Q^I_\alpha, Q^J_\beta\} = Z\epsilon_{\alpha\beta}\epsilon^{IJ},\ \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = Z\epsilon_{\dot\alpha\dot\beta}\epsilon^{IJ}. \tag{2.9}$$

Introduce the annihilation operators

$$b^1_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha + \epsilon_{\alpha\beta}\bar{Q}^2_\beta), \quad b^2_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha - \epsilon_{\alpha\beta}\bar{Q}^2_\beta). \tag{2.10}$$

The only nontrivial anticommutation relations are (Z is assume as real for simplicity)

$$\{b^1_\alpha, (b^1_\beta)^+\} = \delta_{\alpha\beta}(2M+Z), \quad \{b^2_\alpha, (b^2_\beta)^+\} = \delta_{\alpha\beta}(2M-Z). \tag{2.11}$$

From this one sees that the BPS bound is $2M \geq Z$. If $2M \neq Z$, there are 4 creation operators, so the number of states are 16. The helicity content is $(-1, -\frac{1}{2}^4, 0^6, \frac{1}{2}^4, 1)$. On the other hand if $2M = Z$, there are only 2 creation operators, and the number of states are just 4, this short multiplet plays an essential role for $\mathcal{N} = 2$ theory. There are vector multiplet and hypermultiplet as the massless case, we call them BPS multiplet. One can define second helicity supertrace as

$$\Omega = -\frac{1}{2}Tr(-1)^{2J_3}(2J_3)^2, \tag{2.12}$$

$\Omega = 0$ for the long massive multiplet; $\Omega = -1$ for BPS vector multiplet and $\Omega = 1$ for hypermultiplet (these numbers are counting only particles).

The $\mathcal{N} = 1$ lagrangian can be elegantly written using superspace. Besides the space time coordinate, we introduce new fermionic coordinates $\theta, \bar\theta$. The chiral superfield has the following expansion form

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y), \tag{2.13}$$

where $y = x^\mu + i\theta\sigma^\mu\bar{\theta}$. The vector multiplet is expanded in Wess-Zumino gauge as

$$V_\mu = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{2.14}$$

The general lagrangian is encoded by Kahler potential and the Superpotential

$$L = \frac{1}{4\pi}\int d^4\theta K(\bar{\Phi}, \Phi) + \int d^2\theta[W(\Phi) + \frac{1}{32\pi i}\tau(\Phi)TrW_\alpha^2] + c.c. \tag{2.15}$$

Here $\bar{\Phi} = \Phi^+ e^{-2V}$, W_α is the spinor chiral superfield building from V_μ, and the trace is taken in the fundamental representation. In general, this action is non-renormalizable, but it can be used to describe the low energy behavior of a renormalizable theory. $W(\phi)$ is a holomorphic function, this homomorphic property is remarkably useful to study supersymmetric gauge theories.

For the UV renormalizable lagrangian, $\tau = \frac{\theta}{2\pi} + \frac{i4\pi}{g^2}$ is constant and the Kahler potential takes the standard form $K(\bar{\Phi}, \Phi) = Im(\tau)\bar{\Phi}\Phi$. A $\mathcal{N} = 2$ vector multiplet consists of a $\mathcal{N} = 1$ vector multiplet and a chiral multiplet Φ both transforming in the adjoint representation of the gauge group. The lagrangian should be also of the form (2.15) with special superpotential and Kahler potential. Since the fermions in gauge multiplet and fermions in chiral multiplet are the doublet under $SU(2)_R$, there should be no superpotential for the chiral multiplet. One can show that the lagrangian indeed has the $\mathcal{N} = 2$ supersymmetry.

The hypermultiplet decomposes into two chiral multiplets $(Q, \tilde{Q})$,which transform in complex conjugate representations of the gauge group.When the hypermultiplets are added, there will be a superpotential term

$$W = \sqrt{2}\tilde{Q}\Phi Q + m\tilde{Q}Q. \tag{2.16}$$

So for the UV theory with a lagrangian description, the gauge couplings and the mass terms are all the deformation parameters (for the asymptotical free theory, the gauge couplings is transmuted to a dimensional dynamical generated scale).

Going back to most general lagrangian, assume we have only a bunch of $\mathcal{N} = 2$ $U(1)$ vector multiplets. Expanding the lagrangian, to have $\mathcal{N} = 2$ supersymmetry, a necessary condition is to have the same kinetic terms for the fermion in Φ and fermion in vector multiplet. This is possible only if

$$Im(\tau(\phi))_{AB} = \frac{\partial^2 K(\Phi)}{\partial\Phi_A \partial\bar{\Phi}_B}. \tag{2.17}$$

Since τ is a holomorphic function, the Kahler potential can only be linear dependent on $\bar{\Phi}$, and it is completely dependent on the the function τ. The function τ is holomorphic and we can learn a lot about its quantum behavior just from this property, while in general the Kahler potential is a general function and there is little tool available to determine. However, $\mathcal{N} = 2$ supersymmetry connects the kahler potential with the superpotential, and it is possible we can exact solve it. Introduce the function $f(\Phi)$ so that $\tau_{AB}(\Phi) = \frac{\partial^2 F(\Phi)}{\partial\Phi_A \partial\Phi_B}$, then the Kahler potential is $K(\Phi, \bar{\Phi}) = \frac{\partial F(\Phi)}{\partial\Phi_A}\bar{\Phi}_A$, and the lagrangian can be written as

$$L = \frac{1}{4\pi} Im[\int d^4\theta \frac{\partial F(\Phi)}{\partial\Phi_A}\bar{\Phi}_A + \int \frac{1}{2} d^2\theta \frac{\partial^2 F(\Phi)}{\partial\Phi_A \partial\Phi_B} W_\alpha^A W^{B\alpha}]. \tag{2.18}$$

The function $F(\Phi)$ is called prepotential which is what we need to solve for the low energy theory.

The above construction applies to the theory with lagrangian description, however, these are only a very small subset of all possible $\mathcal{N} = 2$ theories. There are other interesting $\mathcal{N}=2$ theory for which no lagrangian description is known. Such theories can be found as the IR limit of those theories with lagrangian description, etc. Later I will discuss a way of constructing those non-lagrangian theories.

1. Six dimensional construction

The property of lower dimensional quantum field theory sometimes is easily understood from higher dimensional theory. Here we introduce a six dimensional construction of four dimensional $\mathcal{N}=2$ theory. Later I will use a different six dimensional construction. The six dimensional theory we use here is a 6d $(1,0)$ theory. In six dimension, a gauge field has 4 degree of freedom and the minimal spinor also has 4 degree of freedom, so it is possible to have a SUSY gauge theory with only gauge fields A_I and minimal spinor λ. The six dimensional lagrangian is

$$L = Tr(-\frac{1}{4}F_{IJ}F^{IJ} + i\bar{\lambda}\Gamma^a D_a\lambda). \tag{2.19}$$

The R symmetry group is $SU(2)_R$. The SUSY transformation is

$$\begin{aligned} \delta A_I &= i\bar{\eta}\Gamma_I\lambda, \\ \delta\lambda &= \frac{1}{2}\Gamma^{IJ}F_{IJ}\eta. \end{aligned} \tag{2.20}$$

Do dimensional reduction, namely, take all the fields independent of coordinate x^4, x^5. Then in four dimensions, the gauge fields split as a gauge field A_μ, and two scalar A_4, A_5, the rotation group $SO(2)=U(1)$ becomes four dimensional $U(1)_R$ symmetry. The complex scalar $\phi = A_4 + iA_5$ has charge 2 under $U(1)_R$. A chiral spinor decomposes into two Weyl spinor in four dimension which is a doublet under $SU(2)_R$ and carrying charge 1 under $U(1)_R$, so in four dimension the field content is exactly the same as the four dimensional $\mathcal{N}=2$ vector multiplet. The action after dimensional reduction is just the four dimensional $\mathcal{N}=2$ theory with only vector multiplet. The four dimensional SUSY algebra has central terms which are just the momentum along x^4, x^5. The particles carrying these central charges are the Kluza-Klein modes, they do not carry the gauge group charge though.

Six dimensional $(1,0)$ theory can also has the hypermultiplet which upon dimensional reduction becomes four dimensional hypermultiplet. So we can engineer a lot of four dimensional $\mathcal{N}=2$ theory from six dimensional theory. The field content is quite similar to the four dimensional theory though. Since six dimensional theory has the same number of supercharges as four dimensional theory, the construction is quite limited, we can only do the dimensional reduction to get a four dimensional theory. The symmetry and the origin of the central charge is more clear from six dimensional theory and the theory is put in a much more compact form.

There are other six dimensional theories which can be realized as the low energy effective theory of world-volume theory of the branes in string theory. For instance, there are six dimensional $(1,1)$ theory which can be derived by dimensional reduction from 10 dimensional SYM theory. These theories are living on Type IIB $D5$ branes. D3 branes can ending on D5 branes and carrying the gauge charge on $D5$ branes, from the point of view of gauge theory on $D5$ branes, D3 branes are monopoles. These are the interesting BPS states we are interested. There are another six dimensional theory called $(2,0)$ theory, this theory is superconformal and does not have a lagrangian description, however, this theory turns out to be extremely useful to engineer four dimensional $\mathcal{N}=2$ gauge theory, we will review these theories in more detail later. These theories are the world volume theory living on $M5$ branes of M theory, and similarly $M2$ branes can end on M5 branes and it is also possible to extract the BPS spectrum of four dimensional theory using those brane construction.

Similarly, it is interesting to see if we can also get four dimensional $\mathcal{N}=2$ theory from $(1,1)$ theory and $(2,0)$ theory. There are more freedoms here and it is possible to engineer a lot of other $\mathcal{N}=2$ theory.

Here we need to break half of supersymmetry down to four dimensions. There are two ways to preserve some SUSY, the first is to find the Killing spinor on the compact two dimensional surface. The only choice for the Riemann surface is then the torus and we are left with $\mathcal{N}=4$ in four dimension. For the curved Riemann surface, to preserve some supersymmetry, one can do a R twisting of the higher dimensional theory. For $(2,0)$ theory, the R symmetry is $SO(5)_R$, one can use a subgroup to twist the theory. Use a subgroup $SO(3)_R\times SO(2)_R$ of $SO(5)$, the holonomy group of the Riemann surface is $SO(2)$, so we can do a partial twist using $SO(2)_R$ and we are left a $SU(2)_R\times U(1)_R$ symmetry which is exactly the R symmetry group of four dimensional theory. The difficulty is that it is hard to see what is the four dimensional theory. Later I will show how we can read four dimensional theory from the information on Riemann surface. For the $(1,1)$ theory, the R symmetry is just $SO(4)$ and the above naive twisting would not give the correct symmetry of four dimensional theory.

B. Seiberg-Witten fibration

Consider first four dimensional $\mathcal{N}=2$ supersymmetric field theories with lagrangian description. The fields of $\mathcal{N}=2$ vector supermultiplet decompose into an $\mathcal{N}=1$ vector multiplet and a chiral multiplet. The $N=1$ vector multiplet consists of fields $\mathcal{A}=(A,\lambda,D)$, while the chiral multiplet has the fields $\Phi=(\phi,\psi,F)$. The vector multiplet has a $SU(2)_R\times U(1)_R$ symmetry. (λ,ψ) transforms as a doublet under $SU(2)_R$ symmetry while the other fields are singlet. The gauge field is a singlet under $U(1)_R$; the fermion doublet has charge 1 and the complex scalar has charge 2. The $U(1)_R$ symmetry is generically anomalous because of quantum effects. The hypermultiplet decomposes into two chiral multiplets $(Q,\tilde{Q})$, which transform in complex conjugate representations of the gauge group.

Let's look at pure $SU(2)$ theory with only a $\mathcal{N}=2$ vector multiplet. The beta function for the gauge coupling is negative and the theory is asymptotical free. The UV lagrangian is

$$L = \frac{1}{4\pi}\int d^4x d^2\theta \tau_0(\Lambda_0) Tr W_\alpha W^\alpha + c.c. + \frac{1}{4\pi}\int d^4x d^4\theta Im\tau_0(\Lambda_0) Tr\bar{\Phi}\Phi, \qquad (2.21)$$

where $\tau_0(\Lambda_0) = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2(\Lambda_0)}$ is the complex gauge coupling, and Λ_0 is the UV cutoff. There is a dynamical generated scale Λ and the $\tau(\Lambda_0)$ can be expressed in terms of Λ. The potential of this theory is

$$V \propto \int d^4x Tr([\phi, \bar{\phi}]^2). \qquad (2.22)$$

This can be most easily seen from above six dimensional construction: on dimensional reduction, $F_{45} = i[A_4, A_5]$, and the kinetic term F_{45}^2 would give the four dimensional potential term if we identify $\phi = A_4 + iA_5$.

The moduli space of vacua can be determined by minimizing the potential: the complex scalar has the expectation value $\phi = a\sigma_3$:

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \qquad (2.23)$$

The gauge invariant coordinate on the moduli space is $u = Tr\phi^2 = 2a^2$. At a generic point of the moduli space, the gauge symmetry is broken to $U(1)$ and this moduli space is called Coulomb branch (there can be other branch when we add matter). There is only one massless $U(1)$ vector multiplet in the low energy theory whose effective interaction is derived by integrating out massive W boson. The low energy effective action at generic point is a free field theory and has the lagrangian

$$L = \frac{1}{4\pi} Im[\int d^4xd^2\theta \frac{1}{2}\frac{\partial^2 F(A)}{\partial A^2} W_\alpha W^\alpha + \int d^4xd^4\theta \frac{\partial F(A)}{\partial A}\bar{A}]. \tag{2.24}$$

To solve the low energy theory is equivalent to determine the prepotential $F(A)$. The effect of integrating out the massive modes is encoded into the gauge coupling $\tau(a)$:

$$\tau(a) = \frac{\partial^2 F(a)}{\partial^2 a}. \tag{2.25}$$

The kahler metric on the moduli space is

$$ds^2 = Im(\tau(a)) da d\bar{a}, \tag{2.26}$$

which must be positive definite.

Define the magnetic variable $a_D = \frac{\partial F(a)}{\partial a}$, the kahler metric on the moduli space becomes

$$ds^2 = -\frac{i}{2}(da da_D - da_D da). \tag{2.27}$$

The metric shows that the local coordinate is not unique and this simple mathematical fact has far reaching physical origin as I will describe later.

Besides the massive electric charged W boson, there are also magnetic charged BPS states whose charges are are given by

$$Z = n_e a(u) + n_m a_D(u). \tag{2.28}$$

The mass for these BPS particles are $M = 2|Z|$. The massive W boson has charge $(2, 0)$ in this normalization. The BPS particle saturates this bound and they are playing a critical role in determining strongly coupled dynamics.

In the large u region of the moduli space, the gauge coupling is small and the perturbative calculation is reliable, the prepotential F receives one-loop correction and instanton corrections which are in general difficult to determine. In the small u region which is the strong coupling region, one does not have any tools to calculate the prepotential. In the semiclassical region, the one-loop result is

$$
\begin{aligned}
a &\cong \sqrt{u/2}, \\
a_D &\cong i\frac{1}{\pi} a \log \frac{a^4}{\Lambda^4},
\end{aligned}
\tag{2.29}
$$

This can not be the whole story since as we approach to the strong coupling region, the kahler metric is not positive definite, so the non-perturbative effects is needed to make the kahler metric positive definite.

Seiberg-Witten [1] found the solution by noting the electric-magnetic duality of the low energy $U(1)$ theory. The essential idea is that the description of the physics is not unique: different sets of variables and different gauge couplings can be used to describe same physics. This can already be seen from the semi-classical formula (2.29): if we go around the infinity, the coordinate change as $a \to -a, \quad a_D \to a - 4$, there is a monodromy acting on (a, a_D) when goes around infinity. In fact, there is a $SL(2, Z)$ duality group acting on the theory which is a generalization of electro-magnetic duality. Under this duality, the gauge coupling $\tau(u)$ underwent a $SL(2, Z)$ transformation and this is the origin of the freedom of using different coordinates (a, a_D) to describe the same physics.

On the u plane, there are two singular points on the strong coupling region of moduli space around which there is a nontrivial monodromy. The physical explanation of these singularities is that there are extra massless fields appearing here; these extra massless particles are the monopoles in the semiclassical region.

The moduli space then looks like that depicted in Figure 1. The complex structure moduli of the torus is identified with the gauge coupling τ. This automatically

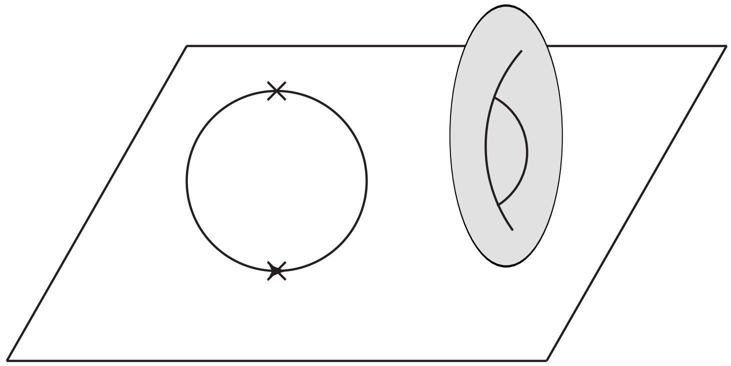

Fig. 1. The Coulomb branch of pure SU(2) gauge theory. There is an auxiliary torus attached at each point. Two singularities appear in the strongly coupled region. There is a circle goes through the singularities representing the marginal stability wall across which BPS particles can decay.

ensures the positiveness of the kahler metric on the moduli space. The complex structure of the moduli is not unique but underwent a $SL(2,Z)$ monodromy when goes around the singularity. At the singularity, the torus degenerates and the generic description of the physics breaks down: extra massless states appear at the singularity.

The solution is nicely encoded into an Seiberg-Witten fibration equation

$$y^2 = \Lambda^2 z^3 + 2uz^2 + \Lambda^2 z. \tag{2.30}$$

For each fixed u, the above equation defines a torus. At $u = \infty, \Lambda, -\Lambda$, the torus degenerates. These are just the singular points on the moduli space. The prepotential is determined by taking a differential $\lambda = \frac{ydz}{z^2}$ and two one cycle A, B on the torus, and the coordinate a and a_D are defined as

$$\begin{aligned} a(u) &= \int_A \lambda \\ a_D(u) &= \int_B \lambda. \end{aligned} \tag{2.31}$$

There is one more issue I want to stress: the BPS spectrum is only piecewise constant on the moduli space. There is a codimension one marginal stability wall across which some of the BPS states will decay. This kind of decay process is essential for the consistency of the Seiberg-Witten picture. Recently there are exciting development in determining the exact BPS spectrum and their wall crossing behaviors [12].

So to describe the IR dynamics of $\mathcal{N}=2$ gauge theory, one need to specify a Seiberg-Witten fibration which is described by a algebraic curve, we also need to specify an Seiberg-Witten differential and the metric can be calculated from this data. The form of the curve and the differential is not unique though. There are singularities on the moduli space at which there are new massless particles appearing. The monodromy around the singularity can also be calculated from the curve, etc. The above curve (2.36) can be put in another form which is useful for us later. Let's define new coordinate $x = y/z$, then the curve becomes

$$x^2 = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}. \tag{2.32}$$

with $\lambda = xdz$.

The above analysis can be generalized to $\mathcal{N}=2$ SU(N) gauge theory with N_f fundamental matters [7, 8]. In this case, the moduli space has complex dimension $N-1$, where $N-1$ is the rank of the gauge group. There is a family of Riemann surface fibred on each point on moduli space with $N-1$ pair of one cycles (A_i, B_i). The Seiberg-Witten curve reads

$$y^2 = P_N(x)^2 - \Lambda^{2N-N_f}\prod_{i=1}^{N_f}(x+m_i), \tag{2.33}$$

where $P_N(x) = x^N + u_2 x^{N-2} + u_3 x^{N-3} + ...u_N$, here $u_2...u_N$ parameterize Coulomb branch. The Seiberg-Witten differential is

$$\lambda = xd\log\frac{P_N(x)-y}{P_N(x)+y}. \tag{2.34}$$

I would like to transform it to a convenient form which is useful later. Define $P_N - y = \Lambda^{N-M} Q_M(x) z$, $P_N + y = \Lambda^{N-M^{'}} Q_{M^{'}}(x)\frac{1}{z}$, where $M + M^{'} = N_f$, $M^{'} \leq M \leq N$. The polynomial $Q_M(x)$ and $Q_{M^{'}}(x)$ satisfy the relation $Q_M(x) Q_{M^{'}}(x) = \prod_{i=1}^{N_f}(x + m_i)$. Change also coordinate $x \to xz$, the Seiberg-Witten curve becomes

$$x^N + \frac{u_2}{z^2} x^{N-2} + \frac{u_3}{z^3} x^{N-3} + ... \frac{u_N}{z^N} = \frac{1}{z^N}[\Lambda^{N-M} Q_M(x) z + \Lambda^{N-M^{'}} Q_{M^{'}}(x) \frac{1}{z}]. \quad (2.35)$$

The Seiberg-Witten differential is $\lambda = xdz$ (there is an extra term which gives the coordinate a, a_D a shift of the flavor charge, we can therefore ignore them). In particular, for the pure SU(N) Yang-Mills theory, $G(x) = \Lambda^N$ This form will make the direct connection to the integrable system later. It is interesting to write some explicit Seiberg-Witten curve for the $SU(2)$ gauge theory with $N_f \leq 3$:

$$\begin{aligned} N_f = 1: \quad x^2 &= \frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{m\Lambda}{z} + \Lambda^2, \\ N_f = 2 \ M^{`} = M = 1: \quad x^2 &= \frac{\Lambda^2}{z^4} + \frac{\Lambda m_1}{z^3} + \frac{u}{z^2} + \frac{\Lambda m_2}{z} + \Lambda^2, \\ N_f = 2 \ M^{'} = 0, M = 2: x^2 &= \frac{\Lambda^2}{z^3} + \frac{u}{z(z-1)} + \frac{m_2^2}{(z-1)^2} + \frac{m_1^2}{z^2}, \\ N_f = 3: x^2 &= \frac{m_1^2}{z^2} + \frac{m_2^2}{(z-1)^2} + \frac{u}{z(z-1)} + \frac{m_3\Lambda}{z} + \Lambda^2. \end{aligned} \quad (2.36)$$

We have made a shift of x coordinate so that term linear in x is vanishing.

The above curves are derived by some physical considerations and guesswork, there is no general method to find the Seiberg-Witten curve given a $\mathcal{N} = 2$ field theories. String theory gives effective methods of constructing Seiberg-Witten curve given certain class of field theories, I will review this construction in next section.

C. String theory construction

String theory provides powerful way to determine the Seiberg-Witten fibration of some gauge theories. I will discuss one construction proposed by Witten [9] in this

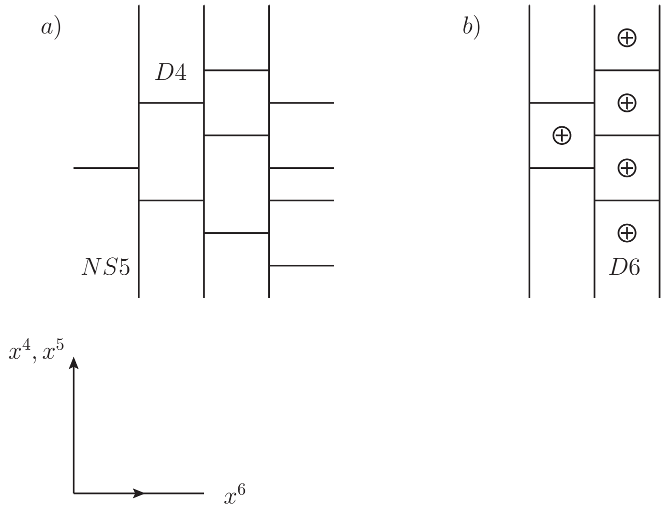

Fig. 2. Left: A Type IIA NS5-D4 brane configuration which gives four dimensional $\mathcal{N} = 2$ superconformal field theory, there are semi-infinite D4 branes on both ends which provide the fundamental hypermultiplets. Right: Instead of semi-infinite D4 branes, $D6$ branes is used to provide fundamental hypermultiplets.

section, since this construction is closed related to recent development of S duality.

A large class of four dimensional $\mathcal{N} = 2$ field theories can be engineered by using Type IIA NS5-D4-D6 brane configurations (for the introduction to string theory, see [30, 31, 32]). The NS5 branes which extend in the direction $x^0, x^1, x^2, x^3, x^4, x^5$, are sitting at $x^7, x^8, x^9 = 0$ and at the arbitrary value of x^6. The x^6 position is only well defined classically. The D4 branes are stretched between the fivebranes and their world volume is in x^0, x^1, x^2, x^3 direction; These D4 branes have finite length in x^6 direction. $D6$ branes extend in the direction $x^0, x^1, x^2, x^3, x^7, x^8, x^9$ can also be added. Two typical brane configurations are depicted in Figure 2.

NS5 brane and D6 branes are heavy, so we can treat these branes as classical object. The dynamics of the system are controlled by the theory on D4 branes, since D4 branes has a finite extent in x^6 direction, and the effective theory is a four dimensional gauge theory. The brane systems preserve four dimensional $\mathcal{N}=2$ supersymmetry, so this gives an effective way of construction four dimensional $\mathcal{N}=2$ theory. There is a U(N) gauge group supporting at each D4 brane segment (the U(1) factor is frozen). Type IIA theory has a hidden coordinate x^10 which is invisible in perturbation theory. In terms of complex coordinate $s=(x^6+ix^{10})/R$ and $v=x^4+ix^5$, the complex gauge couplings are

$$-i\tau_\alpha(v)\cong s_\alpha(v)-s_{\alpha-1}(v), \tag{2.37}$$

here s_α is the position of αth NS5 brane and is a logarithmic function of v, in fact

$$-i\tau_\alpha(v)\cong(2k_\alpha-k_{\alpha-1}-k_{\alpha+1})\ln v, \tag{2.38}$$

here k_α are the number of D4 branes between αth and $\alpha+1$th NS5 Brane. This is the familiar picture of running gauge coupling in four dimension. There are bi-fundamental matter between adjacent gauge groups whose mass parameter are given by position difference of two stacks of D4 branes. The mass of fundamental matter is also represented by the v coordinate of D6 branes. All the UV parameters have very nice geometric interpretation.

There are two different ways to introduce fundamental hypermultiplets to the gauge groups at both ends: This can be achieved by attaching semi-infinite D4 branes as in Figure 2a) or adding D6 branes as in Figure 2b). In this section, brane configurations with semi infinite D4 branes will not be considered. Let's consider a brane configuration with $n+1$ NS5 branes and a total of k_α D4 branes stretched between αth and $(\alpha+1)$th NS5 brane, the gauge group is $\prod_{\alpha=1}^{n}SU(k_\alpha)$, and there are bifundamental hypermultiplets transforming in the representation $(k_\alpha,\bar{k}_{\alpha+1})$; The number of $D6$ branes d_α is constrained by

$$d_\alpha \leq 2k_\alpha - k_{\alpha+1} - k_{\alpha-1}, \tag{2.39}$$

where $k_0 = k_{n+1} = 0$. The restriction on the number of fundamental matter ensures that each node of the quiver theory is either superconformal or asymptotical free.

The Seiberg-Witten curve for this theory is derived by lifting the above configuration to M theory. NS5 brane becomes a M5 brane located at a fix position at x^{10}, D4 branes become also the M5 branes but now wrap on x^{10}. The D6 branes are described by Taub-NUT space with coordinates x^4, x^5, x^6, x^{10}. NS5-D4 brane configurations become a single M5 brane embedded in D6 branes background, and the Seiberg-Witten curve is just the Riemann surface wrapped by this M5 brane. Define coordinate $v = x^4 + ix^5$ and polynomials:

$$J_s = \prod_{a=i_{s-1}+1}^{i_s} (v - e_a), \tag{2.40}$$

where $1 \leq s \leq n$ and $d_\alpha = i_\alpha - i_{\alpha-1}$, e_a is the constant which represents the position of D6 brane. The Seiberg-Witten curve is

$$\begin{aligned} y^{n+1} + g_1(v)y^n + g_2(v)J_1(v)y^{n-1} + g_3(v)J_1(v)^2J_2(v)y^{n-2} \\ +... + g_\alpha \prod_{s=1}^{\alpha-1} J_s^{\alpha-s} y^{n+1-\alpha} + ... + f \prod_{s=1}^{n} J_s^{n+1-s} = 0, \end{aligned} \tag{2.41}$$

here g_α is a degree k_α polynomial of variable v. The Seiberg-Witten differential is given by $\lambda = \frac{vdy}{y}$. The gauge coupling is encoding in the overall coefficient of each polynomial g_α.

The gauge couplings are determined by x^6 positions of the NS5 branes. If the beta functions for all the gauge groups vanish, the asymptotic behaviors of the roots of Seiberg-Witten curve regarded as a polynomial in y determine the gauge couplings.

In large v limit, the roots are $y \sim \lambda_i v^{k_1}$, where λ_i are the roots of the polynomial equation:

$$x^{n+1} + h_1 x^n + h_2 x^{n-1} + ... + h_n x + f = 0. \tag{2.42}$$

In x plane, there are $n+3$ distinguished points, namely $0, \infty$, and λ_i. The choice of λ_i determines the asymptotic distances in coordinate x^6 between fivebranes and hence the gauge coupling constants. The gauge coupling space is the complex structure moduli of the sphere with $n+3$ marked points among which $0, \infty$ are distinguished. Denote the moduli space as $M_{0,n+3;2}$, the fundamental group $\pi_1(M_{0,n+3;2})$ is interpreted as the duality group.

Let's describe one example which illustrates the power of the brane construction. Consider first the case with only two NS5 branes and no D6 branes. This is just pure SU(N) gauge theory, the Seiberg-Witten curve from the general formula (5.43) is

$$f y^2 - (v^N + u_2 v^{N-2} + ... u_N) y + f = 0 \tag{2.43}$$

with $y = e^s$, and I have made a scaling to take this special form. In the large v limit, there are two solutions of y

$$y_1 = \frac{1}{f} v^N, \quad y_2 = f v^{-N}, \tag{2.44}$$

then the gauge coupling reads

$$-i\tau(v) = -2N ln \frac{v}{f^{\frac{1}{N}}}, \tag{2.45}$$

so the dynamical generated scale is $\Lambda^N = f$. Define $y \rightarrow z, x \rightarrow xz$, the above curve becomes

$$x^N + \frac{u_2}{z^2} x^{N-2} + \frac{u_3}{z^3} x^{N-3} + ... \frac{u_N}{z^N} = \frac{\Lambda^N}{z^N}(z + \frac{1}{z}), \tag{2.46}$$

with Seiberg-Witten differential $\lambda = x dz$. This is exactly the same as (2.35).

D. Relation to integrable system

Starting with pure $SU(2)$ gauge theory, the low energy effective action is solved by writing a Seiberg-Witten fibration $X \to U$: there is a torus X_u at each point of the moduli space. The kahler metric can be solved by specifying a meromorphic one form λ on X_u. However, λ is not unique: another differential $\lambda^{'} = \lambda + d\alpha$ will play the same role. It is natural to define a two form $\omega = d\lambda$ which is unique. ω is holomorphic and closed as from the definition. The kahler metric on the moduli space is derived as following: start with $(2,2)$ form $\omega \wedge \bar{\omega}$ and integrate over the fibre, the resulting $(1,1)$ form should be positive definite and defines the kahler metric on the moduli space. Locally if $\omega = \alpha \wedge du$ where α is the one form on the torus X_u, then the metric is positive as long as $\alpha \neq 0$ which means that Ω is non-degenerate. Notice that the restriction of ω on the fibre is zero with this consideration. Therefore, ω has the following property: ω is a holomorphic and closed $(2,0)$ form on X, it is non-degenerate and its restriction on the fibre is zero.

The above characterization of the $(2,0)$ form ω can be generalized to the solution of rank r gauge theory. The vanishing restriction on the fibre of ω means that locally $\omega = \sum_i dx_i \wedge du^i$ where x^i are the coordinates on the fibre. Non-degenerate closed holomorphic $(2,0)$ ω defines a complex symplectic structure on X, the Poisson bracket for the holomorphic functions f and g is

$$\{f, g\} = \sum_{i,j} \omega^{ij} \partial_i f \partial_j g. \tag{2.47}$$

Since locally $\omega = \sum_i dx_i \wedge du^i$, there are r commuting Hamiltonian u_i which satisfy the relation $\{u_i, u_j\} = 0$. This means that (X, ω) defines an algebraically completely integrable Hamiltonian system [4]! The inclusion of the mass term in the lagrangian changes the story a little bit: there are a family integrable systems such that ω

depends linearly on the mass parameter.

Therefore finding the solution of a given gauge theory is equivalent to identifying an integrable system. This is another way of finding solutions of gauge theory: finding an integrable system. However, there is no known explicit way to identify the gauge theory and integrable system. Still, one need some guesswork. Let me give an example of how this works.

The integrable system is the so-called periodic Toda chain, it is a non-relativistic system of $n+1$ points on a circular complex chain, with expotential nearest neighbor interactions, given by

$$H = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 - M^2 \sum_{i=1}^{n+1} e^{x_{i+1}-x_i}, \tag{2.48}$$

with the periodicity condition $x_{n+2} = x_1$. The Hamiltonia can written in an elegant way if we recognize the simple roots of lie algebra A_n and affine lie algebra $A_n^{(1)}$,

$$\begin{aligned} A_n:\ & e_i - e_{i+1} \quad i = 1, ..., n \\ A_n^{(1)}:\ & e_i - e_{i+1} \quad i = 1, ..., n+1, e_{n+2} = e_1. \end{aligned} \tag{2.49}$$

Then the interaction term can be written as $V = M^2 \sum_{\alpha\in\mathcal{R}_*} e^{-\alpha.x}$. This is an integrable system since there is a family of Lax pair by selecting a cartan generators $h = (h_1, ...h_n)$

$$\begin{aligned} L(z) &= p\dot{h} + \sum_{\alpha\in\mathcal{R}_*} M e^{-\frac{1}{2}\alpha.x}(E_\alpha - E_{-\alpha}) + \frac{1}{2}\mu^2 e^{\frac{1}{2}\alpha_0.x})(zE_{-\alpha_0} - z^{-1}E_{\alpha_0}) \\ M(z) &= \sum_{\alpha\in\mathcal{R}_*} M e^{-\frac{1}{2}\alpha.x}(E_\alpha - E_{-\alpha}) + \frac{1}{2}\mu^2 e^{\frac{1}{2}\alpha_0.x}(zE_{-\alpha_0} - z^{-1}E_{\alpha_0}), \end{aligned} \tag{2.50}$$

here z is a spectral parameter. The n maximal commuting integral of motions are $I_i = trL^i, i = 1, ...n$ and one can define a spectral curve

$$\det(x - L(z) = 0. \tag{2.51}$$

The Seiberg-Witten differential is $\lambda = \frac{xdz}{z}$, change variable to $x \to xz$, the spectral curve is

$$x^N + \frac{u_2}{z^2}x^{N-2} + ...u_N = \frac{\Lambda^N}{z^N}(z + \frac{1}{z}), \tag{2.52}$$

with Seiberg-Witten differential $\lambda = xdz$, this is exactly the form (2.35).

E. Compactification to three dimensions

The Seiberg-Witten fibration X looks like just a mathematical tool of solving field theory, however it actually has a physical meaning. Compactify four dimensional theory on a circle with radius R [33]. The corresponding three dimensional theory has $\mathcal{N} = 4$ supersymmetry, X is related to the Coulomb branch of this three dimensional theory.

For the pure three dimensional theory, the effective action on the Coulomb branch consists of r three dimensional $U(1)$ vector multiplet whose bosonic fields are a gauge field and three real scalars. In three dimension, a photon is due to a real scalar, so there are a total of 4 real scalars. The low energy effective theory on the Coulomb branch is therefore a $4r$ dimensional hyperkahler manifold.

When the four dimensional theory is compactified on a circle with radius R, the Coulomb branch is also a hyperkahler manifold M. One would like to know the metric of M. In the large R limit, one can borrow the result from four dimensions. The bosonic part of four dimensional effective action is

$$L = -[\int d^4x \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{4\pi} Im(\tau(u)) du d\bar{u}], \tag{2.53}$$

where $\tau(u) = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$. In three dimensional terms, $A_3 = \frac{b}{\pi R}$ component becomes a real scalar, and the gauge field is also dual to a real scalar σ in three dimension, both scalars take values in S^1, so the target space of the gauge field part is a torus, the effective action for the three dimensional massless fields is

$$\tilde{L} = \int d^3x(\frac{1}{\pi Re^2}|db|^2 + \frac{e^2}{\pi R(8\pi)^2}|d\sigma - \frac{\theta}{\pi}db|^2 + \frac{R}{2}Im(\tau(u))dud\bar{u}). \qquad (2.54)$$

So the total space X of Seiberg-Witten fibration is the Coulomb branch of three dimensional theory at least in the large R limit. However, the Coulomb branch is a hyperkahler manifold. The Seiberg-Witten fibration only involves one complex structure of the hyperkahler manifold, and this complex structure does not depend on the radius R. In general, the exact metric at arbitrary R deviates from our naive form (2.54), though one of its complex structure does not depend on R. What happens for arbitrary R is that the four dimensional monopoles wrapped on the circle S become three dimensional instantons which will correct the naive metric. Those instanton corrections will resolve the singularity to make the metric smooth. In the large u limit, the metric is asymptotically of the form of the metric (2.54), the smoothness of the metric tells us the information of the BPS spectrum as shown in [12].

CHAPTER III

SIX DIMENSIONAL PERSPECTIVE [*]

As reviewed in last chapter, string theory construction and integrable system are two main approaches used to derive the Seiberg-Witten curve for four dimensional $\mathcal{N}=2$ gauge theories. The string theory construction is quite powerful, since the UV gauge theory and its deformation is explicitly given and the IR data is nicely derived by lifting the brane configuration to M theory. The integrable system approach is rather illuminating and may point to some very deep connections, however, it is fairly difficult to establish maps between integrable system and gauge theory.

Given the string theory construction, it is interesting to derive the integrable system for the gauge theory. This can be done by recognizing the brane construction as a new six dimensional construction. Once this six dimensional construction is identified, one can engineer large class of new field theories using this new method. The six dimensional construction has far-reaching application than just giving the integrable system. It is very useful in exploring S-duality property for gauge theory, constructing strongly coupled theory, providing the tools for studying wall crossing behavior; it is also useful to make connections with two dimensional conformal field theory and three dimensional mirror symmetry, etc. It is rather surprising and amusing that so much can be done by going to six dimension!

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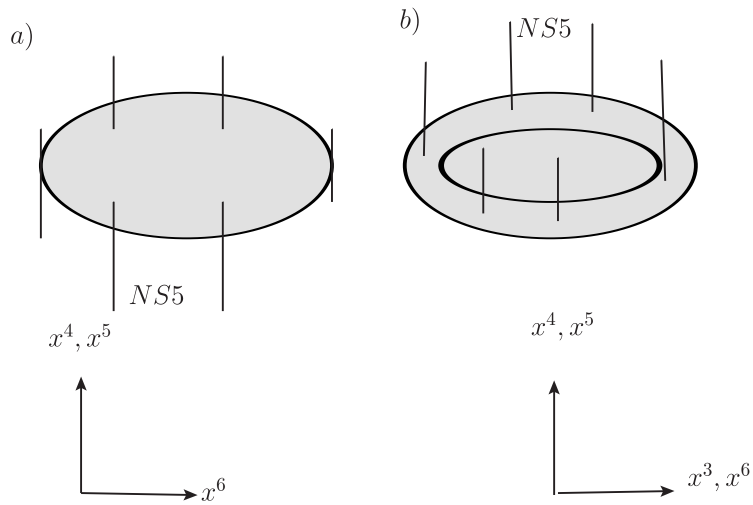

Fig. 3. Left: The electric brane configuration of elliptic model. Right: The magnetic brane configuration of elliptic model.

A. Six dimensional construction from branes: Elliptic model

In fact, the Seiberg-Witten curves derived from the brane construction are quite close to the form derived from the integrable system approach, and it is possible to build a direct relation. In the end, a new six dimensional construction is naturally emerging [34]. This connection to six dimensional theory can be seen by exploring several dualities of the Type IIA brane construction reviewed in Chapter II. We start with the elliptic model studied by Witten [9]. The brane configuration is almost the same as described in last chapter but with x^6 to be compact in present case, see Figure 3. The low energy field theory on this system is a four dimensional $\mathcal{N} = 2$ SCFT called elliptic model. The gauge group is $\mathrm{SU}(k)^n \times \mathrm{U}(1)$. The solution of the low energy theory is also solved by lifting to M theory.

Now we further compactify coordinate x^3 and the system becomes effectively a three dimensional theory. The Coulomb branch of the three dimensional low energy theory is a hyperkahler manifold X with a distinguished complex structure in which it looks like a fibration $\pi : X \to C^r$ with fibers being abelian varieties A_r with complex dimension r [33]. The Coulomb branch is quite difficult to calculate since there is lots of quantum corrections. We want to find a dual theory so that the Coulomb branch of the original theory can be calculated classically. This can be done using various string dualities.

Let's first do a T duality on x^3 coordinate and then do a Type IIB S duality, finally we do another T duality on x^3 coordinate. At the end, we come back to Type IIA configuration. The NS5 brane becomes a IIB NS5 brane under first T duality; S duality turns it into IIB D5 brane, and the second T duality turns it into a IIA D4 brane located at fixed points in x^3, x^6, x^7, x^8, x^9 coordinates, we call it D4$'$ brane; The original D4 branes are not changed. The whole brane configuration becomes $D4-D4'$ system: D4 brane wrapped on x^3, x^6 torus and D4$'$ is sitting at a fixed point of the torus. The gauge theory on D4 branes is a U(k) theory with fundamentals coming from the string stretched between the D4 branes and D4$'$ branes. The Coulomb branch of the original theory is matched to Higgs branch of the dual theory which does not receive quantum corrections and can be calculated classically. The theory on D4 branes are five dimensional Super Yang-Mills (SYM) theory and D4$'$ branes are codimension two impurities on this five dimensional theory. We call the original three dimensional type IIA description electric description and the later type IIA description as magnetic description.

Consider a single NS5 brane, the four dimensional electric theory is just SU(N) Yang-Mills with an adjoint, the gauge coupling is

$$\frac{1}{g^2} = \frac{L}{\lambda}, \tag{3.1}$$

where L is radius of x^6 and λ is the string coupling constant. After a sequence of string dualities[1], the new constants are $\tilde{\lambda} = R^{\frac{3}{2}}\lambda^{-\frac{1}{2}}, \quad \tilde{R} = (R\lambda)^{\frac{1}{2}}, \tilde{L} = L(\frac{R}{\lambda})^{\frac{1}{2}}$, then the complex structure moduli of the torus with coordinates (x^3, x^6) is $\tau = \frac{\tilde{L}}{\tilde{R}} = \frac{L}{\lambda}$, which is just the gauge coupling of the original four dimensional theory, and the gauge coupling of the original four dimensional theory becomes complex structure parameter of the torus in the magnetic description.

Now the difficult question of calculating Coulomb branch of original 3d theory is transformed to the calculation of Higgs branch of five dimensional theory compactified on a torus with several punctures. Since one of the complex structure of the 3d Coulomb branch is independent of R and is equal to the Seiberg-Witten fibration of the corresponding four dimensional theory, the job of calculating Seiberg-Witten curve is reduced to calculate the Higgs branch of Five dimensional theory. The Higgs branch is just the moduli space of Hitchin equation with specific boundary condition at the puncture [35](we will discuss Hitchin equation in great detail later). One of the remarkable property of Hitchin's moduli space is that it is an integrable system in one of distinguish complex structure! In fact, one can similarly write an spectral curve for this integrable system, and this curve is just the Seiberg-Witten curve for four dimensional theory. So at least for the elliptic model, the integrable system corresponding to it is a Hitchin system! The boundary condition at the puncture is due to an extra fundamental matter.

The five dimensional theory living on D4 branes is five dimensional maximal Yang-Mills, which is not UV complete theory though. It is useful to consider its UV completion which is the six dimensional $(2, 0)$ theory. $(2, 0)$ theory is a SCFT and has no lagrangian description. This exotic theory will be explained more later and the property we need here is that when compactified on a circle, it is five dimensional

[1] we set $\alpha^{'} = 1$.

maximal super Yang-Mills theory. $(2,0)$ theory is the low energy effective theory living on M5 brane of M theory. The above lift to six dimension has string theory interpretation.

In the magnetic description, let's go to type IIA strong coupling and this introduces an extra M theory coordinate x^{10}. In the magnetic description, the D4 and $D4'$ branes become $M5$ branes wrapped on x^10, and the three dimensional theory is derived by compactifying six dimensional theory on a three torus with coordinate x^{10}, x^6, x^3, with another set of $M5$ branes wrapping on $x^0, x^1, x^2, x^3, x^{10}$ and intersect transversely on the torus x^3, x^6.

Now let's take the radius $R \to \infty$ in the electric description to recover the four dimensional theory. In the magnetic description, this corresponds to take $\tilde{\lambda} \to \infty$, since the Type IIA string coupling is just the radius of M theory coordinate (by setting $\alpha' = 1$), this means that we take $R_{10} \to \infty$ and the complex structure constant of the torus x^3, x^6 is the gauge coupling of the original theory. X^{10} is now non-compact, and the effective theory is four dimensional and the theory is actually derived by compactifying six dimensional $(2,0)$ theory on a punctured torus parameterized by x^3, x^6. There is a Hitchin equation defined on the punctured torus which is the integrable system for the electric gauge theory. The complex structure of the torus is the gauge coupling constant.

However, the above configuration is really the same as the electric description. This can be seen by lifting the electric description to M theory by introducing M theory circle with radius Λ. The NS5 brane becomes a M5 brane with world-volume $x^0, x^1, x^2, x^3, x^4, x^5$, and D4 brane becomes M5 branes wrapping $x^0, x^1, x^2, x^3, x^6, x^10$. From D4 brane point of view, it is a six dimensional theory $(0,2)$ theory compactified on torus x^6, x^10 whose complex structure is the gauge coupling of the four dimensional theory. This description is exactly the same as the magnetic description. So for four dimensional theory, there is no electric and magnetic description.

The elliptic model example shows that the four dimensional $\mathcal{N} = 2$ gauge theory can be engineered from six dimensional $(2, 0)$ theory on a punctured Riemann surface. The complex structure of the Riemann surface is the gauge coupling of the four dimensional theory. The Hitchin integrable system is living on the punctured Riemann surface which solves the low energy theory of four dimensional theory.

The six dimensional description has an immediate application: since $(2, 0)$ theory is conformal, the compactification only depends on the complex structure moduli of the compact space. Two complex structure related by modular group are equivalent. Four dimensional gauge coupling is realized as the complex structure of Riemann surface, then S-duality is the modular group of the Riemann surface. The S duality of $\mathcal{N} = 4$ super Yang-Mills theory is well known and the elliptic model is just mass-deformed version to $\mathcal{N} = 4$ theory, here we have a geometric understanding. Notice that for the weakly coupled limit, one take one of the coordinate of the torus to infinity, the Riemann surface develops a long thin tube. So the weakly coupled limit corresponds to the degeneration limit of Riemann surface.

For a four dimensional $\mathcal{N} = 2$ SCFT, it is interesting to explore its S duality property. However, it is not known whether the S-dual theory even for the simplest Superconformal field theory, this is rather different from the $\mathcal{N} = 4$ case for which the S-dual theory is known (though it is hard to check). A remarkable result about the S-duality of $\mathcal{N} = 2$ theory is proposed by Argyres and Seiberg. They showed that for SU(3) with six flavors, the dual theory involves a strongly coupled matter along with a weakly coupled SU(2) gauge group. This remarkable observation can actually be realized geometrically as the above elliptic model.

B. The information at the puncture: General case

It is definitely interesting to extend the six dimensional construction to other models engineered using branes. The natural starting point is a linear superconformal quiver gauge theory. There will be D6 branes in the original type IIA description. $D6$ branes are sitting at fixed positions in x^6 coordinate, however, the Seiberg-Witten curve does not depend on those positions so we can move all the D6 branes to infinity. There will be new D4 branes creating when D6 branes cross NS5 branes. This so-called Hanany-Witten effect can be understood by doing a T duality on x^3 and going to type IIB theory; In type IIB theory, such effect has been studied in great detail. The effects of these D6 branes at infinity can be thought of as providing boundary conditions at infinity. When lift to M theory, the D4 branes becomes $M5$ branes wrap on a cylinder and NS5 branes becomes M5 brane intersecting on a single point on the cylinder, the property of this puncture should be the same as the elliptic model case simply from locality consideration. Now we conjecture that there are two extra singularity at two ends of the cylinder and effectively, the theory is engineered by compactifying six dimensional theory $(2,0)$ on a punctured sphere.

There is a Hitchin system defined on Riemann sphere which solves the low energy effective field description. The complex structure of the Riemann surface is the gauge coupling. The weakly coupled description corresponds to the degeneration limit of punctured Riemann surface.

In the six dimensional construction, all the important information is now encoded in the punctures. So how do we know the information on the punctures? The clue is the relation between the integrable system and the Seiberg-Witten curve. One can read the information on the puncture from the Seiberg-Witten curve by using the following identification

$$\det(x - \Phi dz) = \text{Seiber-Witten curve from brane construction} = 0. \tag{3.2}$$

The left-hand side is the spectral curve of Hitchin's integrable system which is determined by the information on the puncture. Seiberg-Witten differential in Hitchin's integrable system description is $\lambda = xdz$.

We consider a four dimensional $\mathcal{N} = 2$ linear quiver gauge theory with a chain of SU groups

$$\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times ... \times \mathrm{SU}(n_{n-1}) \times \mathrm{SU}(n_n) \tag{3.3}$$

and bifundamental hypermultiplets between the adjacent gauge groups and k_a extra fundamental hypermultiplets for $\mathrm{SU}(n_a)$ to make the gauge couplings marginal. The marginality of gauge couplings imposes the constraints on the number of fundamentals:

$$d_a = (n_a - n_{a-1}) - (n_{a+1} - n_a), \tag{3.4}$$

we define $n_0 = 0, n_{n+1} = 0$. Since k_a is nonnegative, we have

$$n_1 < n_2 < ...n_r = .. = n_l > n_{l+1} > ... > n_n. \tag{3.5}$$

We take $n_r = ... = n_l = N$.

The Seiberg-Witten curve is of the general form

$$\begin{aligned} t^{n+1} + g_1(v)t^n + g_2(v)J_1(v)t^{n-1} + g_3(v)J_1(v)^2 J_2(v)t^{n-2} \\ +... + g_\alpha \prod_{s=1}^{\alpha-1} J_s^{\alpha-s} t^{n+1-\alpha} + ... + f \prod_{s=1}^{n} J_s^{n+1-s} = 0, \end{aligned} \tag{3.6}$$

the Seiberg-Witten differential is $\lambda = \frac{vdt}{t}$, and

$$J_s = \prod_{a=i_{s-1}+1}^{i_s} (v - e_a), \tag{3.7}$$

where $1 \leq s \leq n$ and $d_\alpha = i_\alpha - i_{\alpha-1}$, e_a is the constant which represents the position of D6 brane. In the large v limit, all $n+1$ roots t approach to

$$t \to v^{n_1}, \tag{3.8}$$

which means the theory is conformal.

We will identify $t = z, v = xt$ to match Hithcin's integrable system description. The above form is not good for our purpose; if we want to realize it as the spectral curve of a SU(N) Hitchin system, the Seiberg-Witten curve should be of the form

$$\phi_0(t)v^N + \phi_2(t)v^{N-2} + ...\phi_N(t) = 0. \tag{3.9}$$

where $\phi_1(t)$ is a degree $n+1$ polynomial in t and in the large v limit, t is approaching constant which are the roots of $\phi_1(t)$. In fact, we have already done some experience transforming Seiberg-Witten curve to form appropriate for the integrable system, see for instance the formula (2.35). The solution appears to redefine t coordinate so that $t = t^{'} \prod_{i=1}^{n} J_{iL}(x)$, where we split the fundamental matter on each gauge group into the left part and right part $J_i(v) = J_{iL}(v)J_{iR}(v)$, this would not change the Seiberg-Witten differential in an essential way. Substituting this formula into the Seiberg-Witten curve (3.6), after factoring out a common factor, the Seiberg-Witten curve becomes

$$c_0(v)t^{n+1} + \sum_{\alpha=1}^{n} c_\alpha(v)t^{n-\alpha+1} + ...c_{n+1}(v) = 0, \tag{3.10}$$

we use t for t' and the new coefficient reads

$$
\begin{aligned}
c_0 &= \prod_{\beta=1}^{n} j_{\beta L}^{\beta}(v), \\
c_\alpha &= g_\alpha \prod_{\beta=1}^{\alpha} J_{\beta R}^{\alpha-\beta}(v) \prod_{\beta=\alpha+1}^{n} J_{\beta L}^{\beta-\alpha}(v), \quad \alpha = 1, ...n-1, \\
c_n &= g_n(v) \prod_{\beta=1}^{n} J_{\beta R}^{n-\beta}(v), \\
c_{n+1} &= \prod_{\beta=1}^{n} J_{\beta R}^{n+1-\beta}(v).
\end{aligned}
\tag{3.11}
$$

Now the degree of c_α is

$$k_\alpha = n_\alpha + \sum_{\beta=1}^{\alpha}(\alpha-\beta)d_{\beta R} + \sum_{\beta=\alpha+1}^{n}(\beta-\alpha)d_{\beta L}, \tag{3.12}$$

we then have

$$k_{\alpha+1} - k_\alpha = n_{\alpha+1} - n_\alpha + \sum_{\beta=1}^{\alpha} d_{\beta R} - \sum_{\beta=\alpha+1}^{n} d_{\beta L}. \tag{3.13}$$

The above analysis is general and now we focus on the conformal quiver, using the relation $d_\alpha = d_{\alpha L}+d_{\alpha R} = (2n_\alpha - n_{\alpha-1} - n_{\alpha+1})$, so $n_{\alpha+1}-n_\alpha = -d_{\alpha L}-d_{\alpha R}+n_\alpha-n_{\alpha-1}$, then (3.13) simplifies

$$k_{\alpha+1} - k_\alpha = n_1 - \sum_{\beta=1}^{n} d_{\beta L}. \tag{3.14}$$

The difference is a constant and does not depend on α, now to have the generic form (3.9), we must impose the relation $n_1 = \sum_{\beta=1}^{n} d_{\beta L}$. The degree $k_\alpha = k_0$, interestingly, $k_0 = N$, we indeed can have a SU(N) Hitchin system! (Since the above result does not depend how we separate the D6 branes into left and right part, a canonical choice is to move all the D6 branes with $\alpha \leq r$, since there is a relation $\sum_{left} d_\alpha = n_1$, then one can show that $k_0 = N$).

With the canonical choice, the Seiberg-Witten curve becomes the form

$$\phi_0(t)v^N + \phi_1(t)v^{N-1} + ...\phi_n(t) = 0, \tag{3.15}$$

$\phi_0(t)$ has the form

$$t^{n+1} + f_1 t^n + ...f_\alpha t^{n+1-\alpha} + ..f = 0, \tag{3.16}$$

here f_α is the overall coefficient of the polynomial $g_\alpha(v)$. The Roots t_i of $\phi_0(t)$ and and, $0, \infty$ determine the gauge couplings. From the form of the SW curve, $v = xt$ is singular only at those special points, which means the Higgs field is singular at those points, therefore, the number of punctures are exactly $n+3$ and we can form n invariants which are the gauge couplings of the gauge theory. This confirms our previous conjecture that the gauge coupling constants are the complex structure of punctured Riemann sphere.

Let's look at the form of Higgs field Φ at the puncture. We have $\phi_1(t) = \sum m_{iL} t^{n+1} + \sum_{j=1}^{n} m_j t^{n+1-j} + \sum m_{iR}$, by shifting $v \to v - \frac{1}{N}\frac{\phi_1(t)}{\phi_0(t)}$, we can eliminate the v^{N-1} term in (3.15). After shifting, the solution of v near the points t_i is

$$v \sim \frac{1}{(t-t_i)}(M, M, ... - (n-1)M). \tag{3.17}$$

Notice that M is only a complicated function of m_i. To study the solution of v in the limit $t \to \infty$, new coordinate $t^{'} = \frac{1}{t}$ is needed. Then introduce new coordinate $v = xt^{'}$, near $t^{'} = 0$, the solution of x has a regular singularity and the eigenvalue has the form: for the fist d_1 mass parameters, the degeneracy is 1, for the βth mass parameters, the degeneracy is α, we have a total of N roots, since $\sum_{left} \beta d_\beta = N$, finally, the sum of eigenvalues are zero. Similarly, one find that the Higgs field is singular at $t = 0$ and the form of the roots is determined by the right tail.

From above calculation, we show that there are a total of $n+3$ singularities on

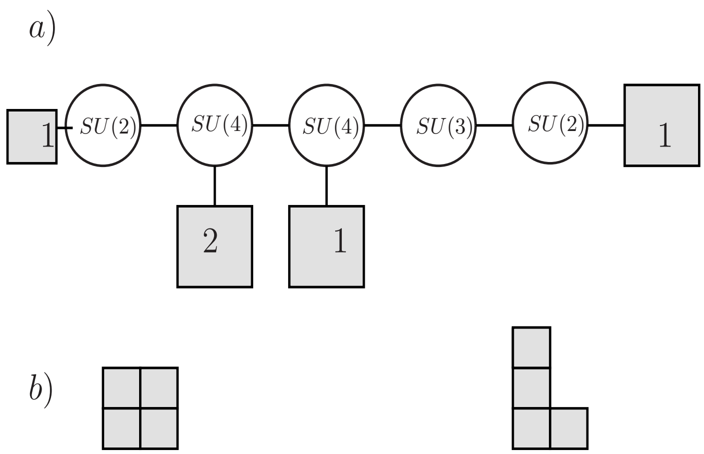

Fig. 4. Top: A $\mathcal{N}=2$ linear quiver with $\mathcal{N}=4$. Bottom: The Young Tableaux associated with left and right tail.

the sphere. $n+1$ has the simple eigenvalues for the Higgs field, while the other two have generic eigenvalues determined by the shape of the quiver. One can label the generic puncture using Young Tableaux.

Let's first consider the left tail. Let us denote $N=n_r=...=n_l$, so that the non-increasing number $(n_a-n_{a-1}), a\leq r$ satisfies the relation $\sum_{a=1}^{a=r}(n_a-n_{a-1})=N$. For the right tail, the non-decreasing number n_a-n_{a+1} starting from $a=n$ also satisfies the relation $\sum_{a=l}^{a=n}(n_a-n_{a+1})=N$. So we associate a Young Tableaux with total boxes N for each tail (see Figure 4). The flavor symmetry of this linear quiver is $U(1)^{n+1}\times\sum_{a=1}^{r}SU(k_a)\times\sum_{a=l}^{n}SU(k_a)$, which can be read explicitly from the quiver diagram. The degeneracy of the eigenvalue of the generic puncture at 0 at ∞ can also be read from Young Tableaux. Notice that there are there are $2n_1-n_2$ column with height 1 and $(2n_i-n_{i-1}-n_{i+1})$ column with height i. Since $d_i=2n_i-n_{i-1}-n_{i+1}$, so we conclude that there is one mass parameter for each column and its degeneracy is the height of that column.

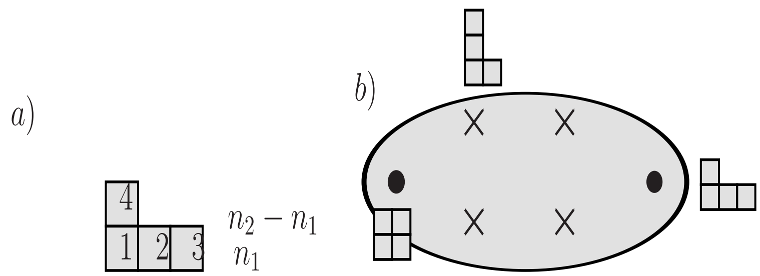

Fig. 5. Left: Young Tableaux associated with the tail in a linear quiver gauge theory with $N = 4$, $p_1 = 1-1 = 0$, $p_2 = 2-1 = 1$, $p_3 = 3-1 = 2$, $p_4 = 4-2 = 2$, the flavor symmetry is SU(2). Right: The punctured sphere for $(0,2)$ A_3 theory compactification, each puncture is labeled by a Young Tableaux.

Actually, given a regular puncture, the flavor symmetry is

$$S(\prod_{l_h>0} U(l_h)). \tag{3.18}$$

where l_h is the number of columns with height h. There is also a natural massless limit for the same puncture, expanding the spectral curve as

$$x^N + \sum_{i=2}^{N} \phi_i(z) x^{N-i} = 0, \tag{3.19}$$

where x is the coordinate on a sphere; and the Seiberg-Witten differential is simply $\lambda = xdz$, here $\phi_i(z)$ is the meromorphic differential.

The orders of the poles are determined from the Young Tableaux by using the formula $p_i = i - s$, where i is the label of the ith box and s is the height of ith box in the Young Tableaux (see Figure 5). The dimension of the space of these meromorphic differentials is given by

$$dimension\ of\ \phi_i = \sum_{punctures\ d=1}^{n+3} p_d^{(i)} + 1 - 2i. \tag{3.20}$$

The parameters of these differential are identified with dimension i operators of the four dimensional theory, i.e. the parameters for the Coulomb branch.

A nature question is if we can put other type of regular singularity at the puncture, it is a little bit surprising that the above situation exhausted all the regular singularity, we will prove this starting from solution of Hitchin's equation.

C. $\mathcal{N} = 2$ SU quiver with USP ends or SU ends with antisymmetric matter

1. Six dimensional construction from orientifold

In the above example, only SU gauge group and fundamental are considered, it is natural to see if the above construction is applicable to other gauge groups and matter contents.

Four dimensional $\mathcal{N} = 2$ superconformal SU field theory with USp ends or SU ends with antisymmetric representations can be derived by adding orientifold six planes to Type IIA D4-NS5 brane system [36]. The solution of the model [37] can be found after lifting the above brane configuration to M theory along the similar line as in [9].

We first consider D4 and NS5 branes system in type IIA theory; We also include two orientifold six planes and 8 D6 branes so that the net RR charges cancels. The k four branes lie along the directions x_0, x_1, x_2, x_3, x_6; we take x^6 coordinate compact. The NS5 branes lie along $x^0, x^1, x^2, x^3, x^4, x^5$ directions. The orientifold six planes extend along $x_0, x_1, x_2, x_3, x_7, x_8 x_9$ directions. It corresponds to the space time transformation

$$h : (x_4, x_5, x_6) \rightarrow (-x_4, -x_5, -x_6), \tag{3.21}$$

together with the world sheet parity Ω and $(-1)^{F_L}$. The D6 branes are parallel to $O6$ planes.

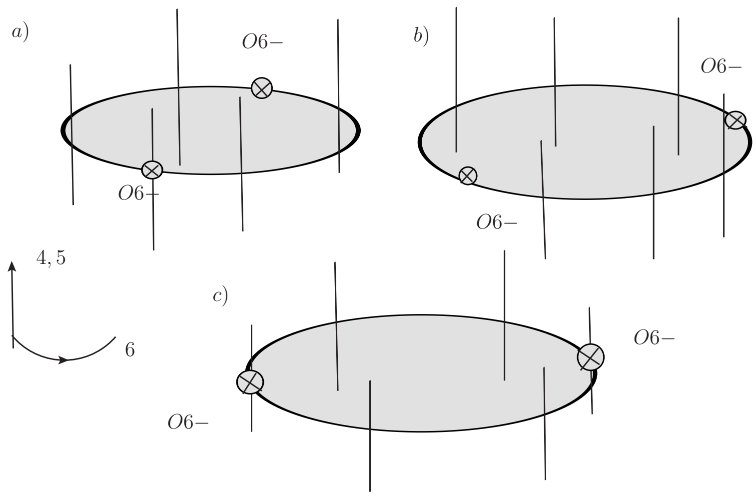

Fig. 6. The tree families of brane configurations in the background of two negatively charged O6-planes. The short vertical lines represent the NS branes, the crossed circles are the orientifold planes. The D6 branes is put in between the NS branes, we omit them in the picture.

There are three main families of $\mathcal{N}=2$ quiver gauge theory with these brane configurations, depending on the positions of the NS branes:

i) The number of NS branes is odd, $N=2r+1$. Only one NS5 brane intersect with the orientifold plane. One typical brane configuration is depicted in Figure 6a. The quiver gauge theory is

$$\mathrm{USp}(k)\times\mathrm{SU}(k)^{r-1}\times\mathrm{SU}(k) \tag{3.22}$$

k must be even since for USp group the rank must be even. We have the bifundamental matter fields between the adjacent group. Two fundamentals are attached at the USp node and we have two fundamentals and one antisymmetric hypermultiplet at the last SU(k) node. The flavor symmetry is $\mathrm{SO}(4)\times\mathrm{U}(1)^r\times\mathrm{SU}(2)\times\mathrm{U}(1)$. The SO(4) flavor symmetry is from the two fundamentals of USp node, and the last $\mathrm{SU}(2)\times\mathrm{U}(1)$ is from the two fundamentals of the SU ends.

Note that the antisymmetric representation of SU(k) is real, so the flavor symmetry of this representation is USp(2) = SU(2). In this paper, however, we do not consider the mass deformation of antisymmetric matter, so we do not include the flavor symmetry associated with it. Use the isomorphism SO(4) = SU(2) $\times$ SU(2), the total flavor symmetry is SU(2) $\times$ SU(2) $\times$ U(1)^r $\times$ SU(2) $\times$ U(1).

ii) The number of NS branes is even. $N = 2r$, and there are no NS branes intersecting the O6-planes. One example is shown in Figure 6b). The quiver gauge theory is

$$\mathrm{USp}(k) \times \mathrm{SU}(k)^{r-1} \times \mathrm{USp}(k) \tag{3.23}$$

We have the bifundamentals between the adjacent group and two fundamentals at the first and last USp gauge factor. The flavor symmetry is SU(2) $\times$ SU(2) $\times$ U(1)^r $\times$ SU(2) $\times$ SU(2).

iii) The number of NS branes is even $N = 2r$. There are two NS branes intersecting with the O6-planes. One configuration is shown in Figure 6c). The quiver gauge theory is

$$\mathrm{SU}(k) \times \mathrm{SU}(k)^{r-1} \times \mathrm{SU}(k) \tag{3.24}$$

Besides the bifundamental matters, we have two fundamentals and one antisymmetric at the first and the last SU factor. The flavor symmetry is U(1) $\times$ SU(2) $\times$ U(1)^r $\times$ SU(2) $\times$ U(1).

When $r = 0$, the above theories are degenerate as

i) USp(k) with a traceless-antisymmetric and 4 fundamentals.

ii) Also a USp(k) with traceless-antisymmetric and 4 fundamentals, this is only for the massless antisymmetric matter, the mass deformation for this matter is not allowed.

iii) SU(k) with 2 antisymmetric hypermultiplets and 4 fundamentals.

The Seiberg-Witten curves for those theories are derived by lifting the Type IIA configuration to M theory [37]. Here we briefly review the derivation. The NS5-D4 brane configuration is lifted to a single M5 brane wrapped on a Riemann surface in $O6-D6$ background. In lifting to M theory, we grow a circular dimension x_{10} with radius R. Define the variables

$$v = x^4 + ix^5, \quad s = (x^{10} + ix^6)/(2\pi R). \tag{3.25}$$

Before orbifolding, the background space is $\tilde{Q} = C \times T^2$. The Z_2 identification of the orientifold is $(v, s) \simeq (-v, -s)$. The M theory background is therefore the orbifold space $Q = \tilde{Q}/Z_2$.

We only need the complex structure of this orbifold background. To do this, we first write an algebraic equation of torus. The torus can be written as an complex curve in the weighted projective space $CP^2_{(1,1,2)}$. $CP^2_{(1,1,2)}$ is defined as the space $(w, x, y)/(0, 0, 0)$ modulo the identification

$$(\lambda\omega, \lambda x, \lambda^2\eta) \simeq (\omega, x, y), \quad \lambda \in C^*. \tag{3.26}$$

The torus is represented as

$$\eta^2 = \prod_{i=1}^{4}(x - e_i\omega), \tag{3.27}$$

where the numbers e_i encode the complex structure τ of the torus in usual way.

The Z_2 automorphism of the torus is $\eta \to -\eta$ with ω and x fixed. The Z_2 identification of the orientifold background becomes $(v, \omega, x, \eta) \simeq (-v, \omega, x - \eta)$. The fixed points are

$$(0, 1, e_i, 0) \quad i = 1, 2, 3, 4, \tag{3.28}$$

we write it in $\omega = 1$ patch.

Let us define Z_2 invariant variables

$$y \equiv \eta v, \qquad z = v^2, \tag{3.29}$$

the orbifolded background Q (without mass deformation for the fundamental matter) is

$$y^2 = z \prod_{i=1}^{4} (x - e_i \omega). \tag{3.30}$$

In the following, we write all the formulas in the patch $\omega = 1$, so the orbifold equation is

$$y^2 = z \prod_{i=1}^{4} (x - e_i). \tag{3.31}$$

The mass deformed (which corresponds to mass deformation to four fundamental matters induced by D6 branes) background is

$$y^2 = z \prod_{i=1}^{4} (x - e_i) + Q(x), \tag{3.32}$$

and

$$Q(x) = \sum_{j=1}^{4} \mu_j^2 \prod_{k \neq j} [(x - e_k)(e_j - e_k)]. \tag{3.33}$$

The Seiberg Witten curve for those field theories is a Riemann surface embedded into above background. We can first write the Seiberg Witten curve for the brane configuration before orbifolding, which is just the elliptic model in [9], and then require the curve invariant under the Z_2 transformation. For the elliptic model, the bifundamental masses satisfy the relation $\sum_\alpha m_\alpha = 0$, so to get the most generic mass-deformed theory, the background is not simply $C \times T_2$ but an affine model. There is no such problem for our model; before orbifolding, the relation $\sum_\alpha m_\alpha$ still applies, however, after orbifolding, the bi-fundamental masses are all independent(the orbifold images of D4 branes have opposite v coordinates, so the bi-fundamental mass for two images are opposite). We do not need to change the background to an affine bundle to allow most generic mass deformation for the bifundamental matters. The situation is different if we want to turn on mass deformation for anti-symmetric matter, the background is an affine bundle. We will not discuss this complication in this paper.

The Seiberg-Witten curve of the above quiver gauge theories without mass deformation is

$$z^n + A(z) + \sum_{s=1}^{r} \frac{B_s(z) + yC_s(z)}{x - x_s} + \sum_{p=1}^{q} \frac{yD_p(z)}{x - e_p} = 0, \quad k = 2n, \tag{3.34}$$

here x_s are positions of NS5 branes which don't intersect with the orientifold; q is the number of NS branes which intersect with the orientifold planes and e_p are positions of NS5 branes stuck at orientifold. This is natural since e_p are fixed points under the orbifold action. $A(z)$ and $B_s(z)$ are polynomials in z

$$A(z) = \sum_{l=1}^{n} A_l z^{n-l}, \quad B_s(z) = \sum_{l=1}^{n} B_{sl} z^{n-l}, \tag{3.35}$$

and C_s and D_p are polynomials in z

$$C_s(z) = \sum_{l=2}^{n} C_{sl} z^{n-l} \quad D_p(z) = \sum_{l=2}^{n} D_{pl} z^{n-l}. \tag{3.36}$$

We also have the constraint:

$$\sum_{s=1}^{r} C_s(z) + \sum_{p=1}^{q} D_p(z) = 0. \tag{3.37}$$

This curve can be derived by first write the Seiberg-Witten curve of elliptic model, and then impose the orbifold invariance and finally express it in terms of orbifold invariant variable. The mass-deformed Seiberg-Witten curve is

$$z^n + A(z) + \sum_{s=1}^{r} \frac{B_s(z) + yC_s(z)}{x - x_s} + \sum_{p=1}^{q} \frac{(y - y_p)D_p(z)}{x - e_p} = 0, \tag{3.38}$$

where $y_p = \sqrt{Q(e_p)}$. $A(z), B(z), C(z), D(z)$ are polynomials in z of order $n-1$.

The Seiberg-Witten differential is given by

$$\lambda = \frac{ydx}{\prod_{i=1}^{4}(x-e_i)}. \tag{3.39}$$

We will rewrite the above curve in a form along the way in [11]. Let's first consider case ii) with two USp ends, which corresponds to $q = 0$. We rewrite the Seiberg-Witten curve in a form which makes the interpretation with the A_{2n-1} theory compactification on a punctured sphere manifest. Expanding the Seiberg-Witten curve in terms of polynomial of z, we have

$$z^n + \sum_{l=1}^{n} \frac{p_r^l(x)}{\Delta'} z^{n-l} + \sum_{l=2}^{n} \frac{y p_{(r-2)}^l(x)}{\Delta'} z^{n-l} = 0, \tag{3.40}$$

here $\Delta' = (x-x_1)....(x-x_r)$ and $p_r^l(x)$ are polynomials with order r; $p_{(r-2)}^l$ are $r-2$ order polynomials. Define $z = \prod_{i=1}^{4}(x-e_i)t^2$, then

$$y = t\prod_{i=1}^{4}(x-e_i) \tag{3.41}$$

The Seiberg-Witten differential becomes

$$\lambda = tdx, \tag{3.42}$$

and the Seiberg-Witten curve is

$$t^{2n} + \sum_{l=1}^{n} \frac{p_r^l(x)}{\Delta' \prod_{i=1}^{4}(x-e_i)^l} t^{2n-2l} + \sum_{l=2}^{n} \frac{p_{r-2}^l(x)}{\Delta' \prod_{i=1}^{4}(x-e_i)^{l-1}} t^{2n-2l+1} = 0. \tag{3.43}$$

With this form, we conclude that this theory can be realized as the six dimensional A_{2n-1} theory compactified on a sphere with r basic punctures $x_1, ...x_r$ (see Figure 7a) for the Young Tableaux and 4 generic punctures $e_i, i = 1, ..4$ with Young Tableaux in Figure 7b). We turn on the surface operators with poles at the punctures:

$$\phi_{2l} = \frac{p_r^l(x)}{\Delta' \prod_{i=1}^{4}(x-e_i)^l} dx^{2l}, \quad \phi_{2l-1} = \frac{p_{r-2}^l(x)}{\Delta' \prod_{i=1}^{4}(x-e_i)^{l-1}} dx^{(2l-1)}. \tag{3.44}$$

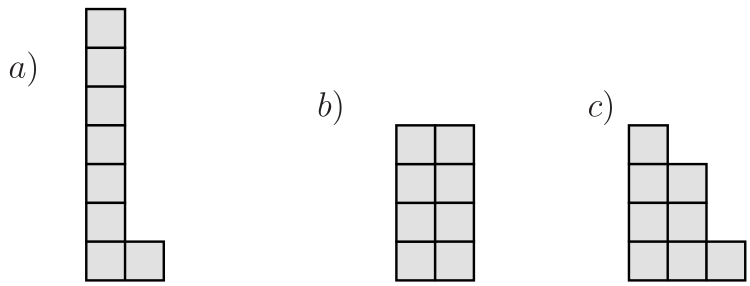

Fig. 7. Young-Tableaux of various punctures. Left: Puncture with $p_i = 1$. Center: puncture with $p_l = \frac{l}{2}$ for even l, $p_l = \frac{(l-1)}{2}$ for odd l. Right: Puncture with $p_l = \frac{l}{2}$ for even l, $p_l = \frac{(l+1)}{2}$ for odd l.

To clarify one point, x is a coordinate on C, and since we do not put any singularity at ∞, we can add a point at ∞ to C and compactify it to a sphere. This does not change the Seiberg-Witten differential and other properties of our model.

Several checks can be made about this conclusion:

a) The moduli space of the sphere with $r+4$ punctures has dimension $r+1$ which can be identified with the coupling constant of gauge groups in the quiver.

b)The various differentials have pole $p_i = 1$ at punctures at x_s , which can be associated with the flavor symmetry U(1), where the Young Tableaux is shown in Figure 7a). The punctures e_i has order $p_l = \frac{l}{2}$ when l is even, $p_l = \frac{l-1}{2}$ when l is odd. This puncture can be represented as a Young Tableaux with two columns of height n in Figure 7b). These poles correspond to SU(2) flavor symmetry. The total flavor symmetry is then $SU(2)^4 \times U(1)^r$, which matches the flavor symmetries read from the quiver diagram.

c) For the differential ϕ_{2l}, the dimension is $4l + r - 2(2l) + 1 = r + 1$, which matches the dimension of the polynomials p^l_r. For differential ϕ_{2l-1}, the dimension is $r - 1$, which also matches the parameters needed for the polynomial p^l_{r-2}.

d) When the mass deformation is turned on, we have the t^{2n-1} term. Do a linear transformation on $t = t' + \alpha$ to eliminate this term. And keep the Seiberg-Witten differential as $\lambda = t' dx$. One can check the residuals of the punctures x_s and e_p have the same patter as determined by the Young Tableaux.

Next, we consider case $i)$ for which only one NS5 brane intersects with the O6 plane. The Seiberg-Witten curve is

$$z^n + A(z) + \sum_{s=1}^{r} \frac{B_s(z) + yC_s(z)}{x - x_s} + \frac{yD_1(z)}{x - e_1} = 0, \quad k = 2n. \tag{3.45}$$

Expand the curve in the polynomial of z and define $z = \prod_{i=1}^{4}(x - e_i)t^2$, the curve becomes

$$t^{2n} + \sum_{l=1}^{n} \frac{p_r^l(x)}{\Delta' \prod_{i=1}^{4}(x - e_i)^l} t^{2n-2l} + \sum_{l=2}^{n} \frac{p_{r-1}^l(x)}{\Delta' \prod_{i=2}^{4}(x - e_i)^{l-1}(x - e_1)^l} t^{2n-2l+1} = 0. \tag{3.46}$$

Similarly, we conclude that this theory can be realized as the six dimensional A_{2n-1} compactified on a sphere with r punctures at x_s and 3 punctures at $e_i, i = 2, 3, 4$, we also have a different puncture at e_1 with Young Tableaux in Figure 7c). The surface operators we turn on are

$$\phi_{2l} = \frac{p_r^l(x)}{\Delta' \prod_{i=1}^{4}(x - e_i)^l} dx^{2l}, \quad \phi_{2l-1} = \frac{p_{r-1}^l(x)}{\Delta' \prod_{i=2}^{4}(x - e_i)^{l-1}(x - e_1)^l} dx^{(2l-1)}. \tag{3.47}$$

Similar checks can be made:

a) The dimension of moduli space of the punctured sphere is $r + 1$ which is identified with the $r + 1$ coupling constants of gauge groups.

b) The flavor symmetries correspond to x_s are U(1), while $e_i, i = 2, 3, 4$ represent flavor symmetry SU(2). The e_1 puncture has pole $p_l = \frac{l}{2}$ for l even, and $p_l = \frac{(l+1)}{2}$ for odd l. This can be represented by the Young Tableaux in Figure 7c). The flavor symmetry of this puncture is U(1). Therefore, the total flavor symmetry is

$\mathrm{U}(1)^r \times \mathrm{SU}(2)^3 \times \mathrm{U}(1)$, Which matches our counting from the quiver diagram. Note that the Young Tableaux for the U(1) from the two fundamentals on the SU ends is different from the U(1) punctures for the bi-fundamental matter.

c) The dimension of ϕ_{2l} is $r+1$, and ϕ_{2l-1} has dimension r, which matches the parameters needed for the polynomial $p_r^l(x)$ and $p_{r-1}^l(x)$.

d) The flavor symmetry can be checked from the mass deformed theory.

Finally, let's consider the quiver in case *iii*); the Seiberg-Witten curve can be written as

$$t^{2n}+\sum_{l=1}^{n}\frac{p_r^l(x)}{\Delta'\prod_{i=1}^{4}(x-e_i)^l}t^{2n-2l}+\sum_{l=2}^{n}\frac{p_{r-1}^l(x)}{\Delta'\prod_{i=3}^{4}(x-e_i)^{l-1}(x-e_1)^l(x-e_2)^l}t^{2n-2l+1}=0. \tag{3.48}$$

Similarly, this theory can be written as the six dimensional A_{2n-1} theory compactified on Riemann surface with punctures e_i and x_s. The surface operators we turn on are

$$\phi_{2l}=\frac{p_r^l(x)}{\Delta'\prod_{i=1}^{4}(x-e_i)^l}dx^{2l},\quad \phi_{2l-1}=\frac{p_{r-1}^l(x)}{\Delta'\prod_{i=3}^{4}(x-e_i)^{l-1}(x-e_1)^l(x-e_2)^l}dx^{(2l-1)}. \tag{3.49}$$

One can check along the similar line that this is the correct interpretation.

2. Some special examples

We want to mention some special examples which are of later interest for us. We first analyze SU(2n) with two-antisymmetric matter and four fundamentals, this corresponds to $r=0, q=2$. The Seiberg-Witten curve is

$$0=t^{2n}+\sum_{l=1}^{n}\frac{A_l}{\sum_{i=1}^{4}(x-e_i)^l}t^{2n-2l}+\sum_{l=2}^{n}\frac{D_l}{\sum_{i=3}^{4}(x-e_i)^{l-1}(x-e_1)^l(x-e_2)^l}t^{2n-2l+1}. \tag{3.50}$$

So this theory can be represented as the A_{2n-1} theory compactified on a sphere with four punctures, two of which have the form as Figure 7a), and two of which have the form as Figure 7b).

We then study the quiver gauge theory corresponding to $r = 1, q = 0$, the quiver gauge theory is USp($2n$) $\times$ USp($2n$). The flavor symmetry in this case is SU(2)^4 $\times$ SU(2). The last SU(2) comes from the bifundamental matter which now furnish a real representation of quiver theory. Naively, we identify this theory as A_{2n-1} compactified on a sphere with four punctures e_i and one basic puncture x_1. The manifest flavor symmetry from this representation is SU(2)^4 $\times$ U(1).

Finally, we consider the quiver corresponding to $r = 0, q = 1$, this is a USp($2n$) theory with four fundamental and one-antisymmetric hypermultiplet. The Seiberg-Witten curve is

$$t^{2n} + \sum_l \frac{p_l}{\prod_{i=1}^4 (x - e_i)^l} t^{2n-2l} = 0. \tag{3.51}$$

This theory is represented as A_{2n-1} theory compactified on sphere with four identical puncture with SU(2) flavor symmetry. Combined with the permutation symmetry of this four identical punctures, we expect that this theory has the $SL(2, Z)$ duality. Notice that the above curve can be written as

$$(t^2 + \frac{q}{\prod_{i=1}^4 (x - e_i)})^n = 0. \tag{3.52}$$

It is amusing to note that for SU(2) theory with four foundamentals, the Seiberg Witten curve is

$$t^2 + \frac{q}{\prod_{i=1}^4 (x - x_i)} = 0. \tag{3.53}$$

So the Seiberg-Witten curve for USp(2n) theory with four fundamentals and one traceless anti-symmetric representation is tensor product of that of SU(2) theory with four fundamentals.

CHAPTER IV

GENERALIZED SUPERCONFORMAL QUIVER GAUGE THEORY: REGULAR SINGULARITY *

In this part, we take the six dimensional $(2,0)$ theory as a starting point and define our theory as the compactification this 6d theory on a Riemann surface with punctures and try to learn as much as we can: gauge group and matter content, S-duality, etc. The Hitchin's equation can be derived by a five dimensional description as derived in last chapter. There is actually another way to see the appearance of Hitchin's equation: The manifold six dimensional field theory living is $\Sigma \times M_4$, where M_4 is a four manifold and usually taken as R^4. We can actually study $N=2$ theory on more general manifold like $M_4 = T^2 \times C$, and get an effective field theory on C. On the other hand, the same system can be seen as first compactifying on T^2 and then on Σ. In the first step, we get a $\mathcal{N}=4$ theory, and in the second step, surface operators of $\mathcal{N}=4$ theory are inserted at the puncture. One need twist $\mathcal{N}=4$ theory to define a supersymmetric theory on a curved Σ. There are actually more than one way to twist the theory; For the twist needed to describe $\mathcal{N}=2$ gauge theory, we get Hitchin's equation on Σ. The twisting is well studied in the context of gauge theory description of Geometric Langlands GL program [38]. I will explain why this so-called GL twist is the correct one for our use.

One starts with a six dimensional $(2,0)$ SCFT. This exotic theory has no lagrangian description. It is the low energy effective field theory living on M5 brane.

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There are three characteristic properties of this theory: 1) It has a ADE classification; 2) When compactified on a circle, it becomes five dimensional maximal supersymmetric theory; 3) It has a Coulomb branch. The second property is used in our derivation of the six dimensional construction. For the A_n type 6d theory, it is just the worldvolume of M5 brane on the flat space-time. The bosonic content is a self-dual two form and five scalars, those five scalars parameterize the motion in the 5 transverse dimension in M theory. The R symmetry is $SO(5)$ which rotates five scalars in a standard way, while the two form fields are singlet.

The important symmetry group is then $SO(1,5) \times SO(5)_R$, the supercharge and scalar transform as

$$Q = 4 \bigotimes 4, \qquad \Phi = 1 \bigotimes 5. \tag{4.1}$$

To get a four dimensional supersymmetric theory, one need to twist the theory on the Riemann surface Σ. The twisting is really combining spin connection $SO(2)_s$ Σ and $SO(2)_R$ to define a new spin connection $SO(2)_s^{'}$. So we split the R symmetry as $SO(3)_R \times SO(2)_R$ and take the diagonal embedding. Before twisting, the supercharge transforming under $SO(1,3) \times SO(2)_s \times SO(3)_R \times SO(2)_R$ as

$$Q = (2_{\frac{1}{2}} + 2^{'}_{-\frac{1}{2}}) \bigotimes (2_{\frac{1}{2}} + 2_{-\frac{1}{2}}). \tag{4.2}$$

We twist the theory using $SO(2)_s^{'} = SO(2)_s - SO(2)_R$, then the supercharge transforms under the symmetry $SO(1,3) \times SO(2)_s^{'} \times SO(3)_R$ as

$$(2 \bigotimes 2)_0 + (2 \bigotimes 2)_1 + (2^{'} \bigotimes 2)_{-1} + (2^{'} \bigotimes 2)_0, \tag{4.3}$$

where the subscript is the charge under $SO(2)_s^{'}$, The preserved supercharge is $(2 \bigotimes 2)_0$ and its conjugate $(2^{'} \bigotimes 2)_0$, which is exactly four dimensional $N = 2$ Supersymmetry.

For the scalar, it transforms under $SO(1,3) \times SO(2)_s \times SO(3)_R \times SO(2)_R$ as

$$1 \bigotimes 5 = 1_0 \bigotimes (3_0 + 1_1 + 1_{-1}). \tag{4.4}$$

After twisting, it transforms under $SO(1,3) \times SO(2)'_s \times SO(3)_R$ as

$$(1 \bigotimes 3)_0 + (1 \bigotimes 1)_1 + (1 \bigotimes 1)_{-1}, \tag{4.5}$$

so there is a complex one form living at the Riemann surface. One also get gauge fields on Σ from the two form.

Now let's look at the same system from $\mathcal{N} = 4$ point of view. There are more than one way to twist the theory on Σ, however, to get a one form scalar on the Riemann surface. The natural choice is the twist appearing in the context of Geometrical Langlands problem. The BPS equation on the Riemann surface is the Hitchin's equation.

A. Review of Hitchin's equation

The four dimensional theory is constructed by compactifying six dimensional theory on a punctured Riemann surface and flowing to IR. The Hitchin's equation is defined on this Riemann surface

$$\begin{aligned} F_A - \phi \wedge \phi = 0, \\ d_A \phi = 0, \quad d_A * \phi = 0, \end{aligned} \tag{4.6}$$

where ϕ_μ and A_μ are two vector fields defined on Riemann surface [39, 40]. The moduli space of solutions of Hitchin's equation M_H is a Hyperkahler manifold, which can be proved by using hyperkahler quotient method. The moduli space has dimension $6g-6$.

A hyperkahler manifold has three complex structures I, J, K satisfying the quaternion relation $I^2 = J^2 = K^2 = 1, IJ = -JI, etc.$ There are three kahler forms $\omega_p = -(gI, gJ, gK), p = I, J, K$. ω_I is a $(1,1)$ form in complex structure I. Define the complex symplectic form $\Omega_I = \omega_K + i\omega_K, etc$, it is a $(2,0)$ form in complex structure I. An isometry G is said to be triholomorphic if $L_X\omega_p = 0$, where X is the vector field generating the symmetry and L_X is the Lie derivative. The momentum map $\vec{\mu}^X$ with respect to ω_p is defined as $X^i\vec{\omega}_{ij} = \vec{\mu}^X_{,j}$.

Given a hypekahler manifold with triholomorphic isometry, one can perform the hyperkahler quotient and the quotient space is $\vec{\mu}^{-1}(0)/G$. In general, one can choose a triplet of central elements $\vec{\zeta}$, and take the quotient $\vec{\mu}^{-1}(\vec{\zeta})/G$.

Let M denote the space of all the fields A and ϕ, one can define a flat metric on it

$$ds^2 = -\frac{1}{4\pi}\int_\Sigma d^2zTr(\delta A_z \bigotimes \delta A_{\bar{z}} + \delta A_{\bar{z}} \bigotimes \delta A_z + \delta\phi_z \bigotimes \delta\phi_{\bar{z}} + \delta\phi_{\bar{z}} \bigotimes \delta\phi_z), \quad (4.7)$$

by choosing a complex structure on Σ, we write $A = dzA_z + d\bar{z}A_{\bar{z}}, \phi = dz\phi_z + d\bar{z}\phi_{\bar{z}}$. The three complex structures are defined on M as (here their action are written on the one form)

$$\begin{aligned} I^t(\delta A_{\bar{z}}) = i\delta A_{\bar{z}}, \quad I^t(\delta\phi_z) = i\delta\phi_z, \quad I^t(\delta A_z) = -i\delta A_z, \quad I^t(\delta\phi_{\bar{z}}) = -i\delta\phi_{\bar{z}}, \\ J^t(\delta A_{\bar{z}}) = -\delta\phi_{\bar{z}}, \quad J^t(\delta A_z) = -\delta\phi_z, \quad J^t(\delta\phi_{\bar{z}}) = \delta A_{\bar{z}}, \quad J^t(\delta\phi_z) = \delta A_z, \\ K^t(\delta A_{\bar{z}}) = -i\delta\phi_{\bar{z}}, \quad K^t(\delta\phi_{\bar{z}}) = -i\delta A_{\bar{z}}, \quad K^t(\delta A_z) = i\delta\phi_z, \quad K^t(\delta\phi_z) = i\delta A_z. \quad (4.8) \end{aligned}$$

Choose a complex structure on Σ, we have fields $A_z, A_{\bar{z}}, \phi_z, \phi_{\bar{z}}$. The kahler form have the similar form:

$$\omega_I = \frac{i}{2}\int dz^2 Tr(\delta\phi_{\bar{z}} \wedge \delta\phi_z - \delta A_{\bar{z}} \wedge \delta A_z),$$
$$\omega_J = \frac{1}{2}\int dz^2 Tr(\delta\phi_{\bar{z}} \wedge \delta A_z + \delta\phi_z \wedge \delta A_{\bar{z}}),$$
$$\omega_K = \frac{i}{2}\int dz^2 Tr(\delta\phi_{\bar{z}} \wedge \delta A_z - \delta\phi_z \wedge \delta A_{\bar{z}}). \tag{4.9}$$

The gauge transformation acts as $\delta A = -D\epsilon$, $\delta\phi = [\epsilon, \phi]$, where A and ϕ is the one form on Riemann surface. The momentum map can be derived similarly:

$$\mu_I = F_{\bar{z}z} + [\phi_z, \phi_{\bar{z}}] = 0. \tag{4.10}$$

The complex moment map is

$$\nu_I = \frac{\partial\phi_z}{\partial\bar{z}} + [A_{\bar{z}}, \phi_z] = 0. \tag{4.11}$$

These are just the Hitchin's equation. We also define $\Omega_I = \omega_J + i\omega_K$ and its permutation they have the following form

$$\Omega_I = \frac{1}{\pi}\int dz^2 Tr\delta\phi_z \wedge \delta A_{\bar{z}},$$
$$\Omega_J = -\frac{i}{4\pi}\int_c Tr\delta A \wedge A,$$
$$\Omega_K = -\frac{i}{2\pi}\int d^2z Tr(deltaA_{\bar{z}} \wedge \delta A_z - \delta\phi_{\bar{z}} \wedge \delta\phi_z - \delta\phi_{\bar{z}} \wedge \delta A_z - \delta\phi_z \wedge \delta A_{\bar{z}}). \tag{4.12}$$

Complex structure J does not depend on the complex structure of the Riemann surface! We now explain one of the main result of Hitchin that in complex structure I, the moduli space M_H is an integrable system. First, there is a Hitchin's fibration, for the case SU(2), the base is defined as the quadratic Casmiir operator $\omega = Tr\phi^2$. The space of quadratic differential on the genus g Riemann surface has dimension $3g-3$. The Hitchin fibration is defined as the map $\pi : M_H \to B$ obtained as mapping

(E, ϕ) to $\omega = Tr\phi^2$. The Hamiltonian is defined as the linear function and has number $3g - 3$ on B which is the correct number for an integrable system (for a 2n dimensional complex symplectic space, there are n commuting Hamiltonian to make it an integrable system). Moreover, one can define a spectral curve as we did for some of the integrable system as

$$\det(x - \Phi dz) = 0. \tag{4.13}$$

where Φ is the holomorphic part of the Higgs field.

It is worth pointing out that in complex structure I, each point on the moduli space parameterizes a higgs bundle, while in complex structure J, each point represents a complex flat connection on Riemann surface.

B. Regular singular solution to Hitchin's equation

We have just considered moduli space of smooth solution to Hitcin's equation. There are also singular solutions which are very useful to us. The most general scale-invariant and rotation-invariant singular solution [41, 42] is

$$\begin{aligned} A &= a(r)d\theta + f(r)\frac{dr}{r}, \\ \phi &= b(r)\frac{dr}{r} - c(r)d\theta. \end{aligned} \tag{4.14}$$

$f(r)$ can be set to zero by a gauge transformation and after introducing a new variable $s = -\ln r$, Hitchin's equation becomes Nahm's equations:

$$\begin{aligned} \frac{da}{ds} &= [b, c], \\ \frac{db}{ds} &= [c, a], \\ \frac{dc}{ds} &= [a, b]. \end{aligned} \tag{4.15}$$

Let's first take SU(2) gauge group. These equations are solved by setting a, b, c

to constant α, β, γ of the Lie algebra of SU(2), and they must commute and we can conjugate them to lie algebra of a maximal torus of SU(2). The resulting solution is

$$
\begin{aligned}
A &= \alpha d\theta + ..., \\
\phi &= \beta \frac{dr}{r} - \gamma d\theta + ...
\end{aligned}
\tag{4.16}
$$

We ignore possible terms which are less singular than the terms presented above. The moduli spca of Hitchin's equation with these boundary conditions are basically the same as the smooth case: the moduli space is a hyperkahler manifold, it is an integrable system in complex structure I, etc. In particular, one can define an spectral curve in complex structure I.

We also want to know the behavior of the solution when $\alpha, \beta, \gamma \to 0$. One may think that there is no singularity at all. This is not the case if we note that we may have less singular terms to the equation. When $\alpha, \beta, \gamma \to 0$, those less singular terms play dominant role.

Indeed, we do have less singular solution to Hitchin's equation, the Nahm's equations can be solved by:

$$
a = -\frac{t_1}{s + 1/f}, \quad b = -\frac{t_2}{s + 1/f}, \quad c = -\frac{t_3}{s + 1/f}, \tag{4.17}
$$

where $s = -\ln r$ and $[t_1, t_2] = t_3$ and cyclic permutation thereof which are the usual commutation relations for SU(2) lie algebra. A convenient basis for SU(2) is

$$
e_1 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{4.18}
$$

The choice of f spoils conformal invariance, but it is not natural to make a choice, since then the derivative of A and ϕ with respect to f is square-integrable. So this solution with f allowed to fluctuate is conformal invariant. The advantage of including this parameter is that when $f=\infty$, we get the trivial solution. Combined the previous discussion, the second type of solution can be thought of the zero limit of the first type.

The fact that the second type of solution is a limit of the first type of solution can also be seen by studying moduli space of Hitchin's equation in complex structure J. In complex structure J, solution of Hitchin's equation describes a flat $SL(2,C)$ bundle. It is important to study the monodromy of the flat connection. Define complex-valued flat connection $\mathcal{A}=A+i\phi$ taking value in $SL(2,C)$, the monodromy is $U=P\exp(-\int_l \mathcal{A})$ where l is the contour surrounding the singularity. This monodromy characterizes the singular behavior of the solution. The curvature $\mathcal{F}$ defined as $\mathcal{F}=d\mathcal{A}+\mathcal{A}\wedge\mathcal{A}$ is equal to zero due to Hitchin's equation, so the monodromy calculates as above is independent of the contour we choose. Define $\zeta=\alpha-i\gamma$, the monodromy for our first set of solutions (4.16) is

$$U=\exp(-2\pi\zeta). \tag{4.19}$$

The monodromy of another solution (4.17) is

$$U^{'}=\exp(-2\pi(t_1-it_3)/(s_1+1/f)). \tag{4.20}$$

The conjugacy class of this matrix is independent of s_1 due to the property that (t_1-it_3) can be taken as up triangular form. We choose a basis in which $t_2=e_1$, $t_1=e_3$, and $t_3=e_2$.

Indeed, what's relevant is the conjugacy class for the monodromy. Let's denote the conjugacy class for U as C_ζ, and the conjugacy class for $U^{'}$ is a union of C_0 and $C^{'}$, where C_0 is the conjugacy class for identity and $C^{'}$ is the conjugacy class for unipotent orbit. It can be shown that as $\zeta\to 0$, C_ζ approaches the union of C_0 and $C^{'}$. This also indicates that the second set of solutions is a limit of the first set of solutions.

The physical interpretation is that the $\alpha, \beta, \gamma \to 0$ is the massless limit of the gauge theory. These facts are in agreement with our analysis of Seiberg-Witten curve in last chapter. The mass term is the eigenvalue of $\beta + i\gamma$. There is an important check for this fact. As pointed out by Seiberg-Witten, when mass are included, the symplectic form should depend linearly on the mass term. In our case, the symplectic form in complex structure I indeed depend linearly on $\beta + i\gamma$, this justifies the identification of the mass term with the eigenvalue of matrices $\beta + i\gamma$.

For the SU(N) case, let's first give a short introduction to relevant mathematical results on lie algebra structure, an readable book for physicists is [43]. Since the pole of holomorphic part of the Higgs field is taking value in sl_n, we need to consider the structure of sl_n instead of $\mathfrak{su}(n)$.

If G is a reductive group over C, g its Lie algebra, we study the adjoint action of G on g:

$$\mathcal{O}_X := G_{ad}.X = \{\phi(X) | \phi \in G_{ad}\}. \tag{4.21}$$

The orbits of this action are the conjugacy classes or adjoint orbits.

A semisimple element U of the lie algebra is an element which is diagonizable, a nilpotent element $U^{'}$ is satisfying the relation ${U^{'}}^{n} = 0$, where n is an integer. A conjugacy class $\mathcal{O}_X$ is semisimple if and only if $\mathcal{O}_X = \mathcal{O}_U$; while a conjugacy class $\mathcal{O}_X$ is nilpotent if and only if $\mathcal{O}_X = \mathcal{O}_{U'}$.

A regular semisimple element in Lie algebra can be defined as follows. The characteristic polynomial of a matrix X in sl_n is

$$\Omega(X) = det(t - X). \tag{4.22}$$

We can expand it as

$$\Omega(X) = \sum_{0\leq i\leq m} (-1)^i p_i(X) t^{n-i}. \tag{4.23}$$

p_1 is zero since $trX = 0$. A semisimple element is called regular semisimple if $p_l \neq 0, l \geq 2$. In particular, this meas that the diagonal elements are all different. For sl_2 case , we only have the regular semi-simple orbit while for sl_n case other options are possible.

There are infinite number of semisimple conjugacy classes and we have only finite number of nilpotent conjugacy classes in sl_n algebra. The nilpotent elements of the sl_n lie algebra are labeled by partitions of n and can be put into standard form. Introduce a partition of n satisfy the conditions:

$$d_1 \geq d_2 \geq .. \geq d_k > 0 \quad and \quad d_1 + d_2 + ... + d_k = n. \tag{4.24}$$

We label this partition as $d = [d_1, d_2, ..d_k]$. We can construct Young Tableaux associated with this partition as shown in Figure 8b). We can also construct a dual partition d^t of d. The first row of d^t is the first column of d, and the second row of d^t is the second column of d, and so on. There is another characterization for the dual partition: the parts of d^t is given by the following formula:

$$s_i = \{j | d_j \geq i\}, \tag{4.25}$$

s_i equals the maximal index j so that $d_j \geq i$. We also draw a Young Tableaux of the dual partition in Figure 8b).

Each nilpotent element is labeled by a partition of n. It can be put into a form using only Jordan block. The Jordan block is defined as: given a positive integer i,

$a)$ $b)$

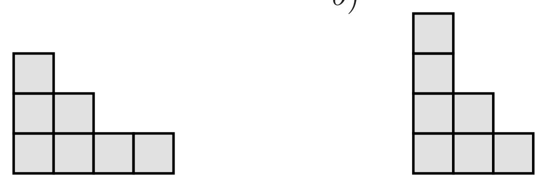


Fig. 8. Left: Young Tableaux of one partition $[4, 3, 1]$ of sl_8. Right: The Young Tableaux for transpose partition [3,2,1,1] of (a).

we construct the $i \times i$ matrix

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{4.26}$$

This matrix is called the elementary Jordan block of type i.

Now the nilpotent element of partition d has the following form:

$$n = \begin{pmatrix} J_{d_1} & 0 & 0 & \ldots & 0 \\ 0 & J_{d_2} & 0 & \ldots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \ldots & J_{d_k} \end{pmatrix}, \tag{4.27}$$

where J_{d_i} is the Jordan block with dimension d_i. The dimension of this nilpotent orbits is given by

$$dim(\mathcal{O}_X) = n^2 - \sum_i s_i^2 = n^2 - 1 - (\sum_i s_i^2 - 1). \tag{4.28}$$

Using the formula $dim(\mathcal{O}_X) = dim(g) - dim(g^X)$, we have $dim(g^X) = (\sum_i s_i^2 - 1)$ and g^X is the centralizer of X, namely the set of elements of lie algebra which commute with g. The maximal dimension occurs when the partition is $d = [n]$, and we call it principal orbit; when $n = [2, 1, 1, ...1]$, the nilpotent orbit has the minimal dimension, we call it minimal orbit.

1. Massless theory

After introducing those mathematical results, let's go back to Hitchin's equation and try to find singular solutions to the equation so that the holomorphic part of the Higgs field has simple pole at the singularity and the residue is an sl_n nilpotent element. In sl_2 case, such a solution is found 4.17; A similar solution for sl_n is constructed using homomorphism between sl_2 and sl_n which involves a nilpotent element of the sl_n algebra.

We introduce a different basis for sl_2 lie algebra:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.29}$$

In this basis the nilpotent element is given by X, they satisfy the commutation relation:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \; and \; [X, Y] = H. \tag{4.30}$$

For an integer $r \geq 0$, we can define a map

$$\rho_r : sl_2 \to sl_{r+1}, \tag{4.31}$$

via

$$\rho_r(H) = \begin{pmatrix} r & 0 & 0 & ... & 0 & 0 \\ 0 & r-2 & 0 & ... & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & ... & -r+2 & 0 \\ 0 & 0 & 0 & ... & 0 & -r \end{pmatrix},$$

$$\rho_r(X) = \begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix},$$

$$\rho_r(Y) = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ \mu_1 & 0 & 0 & ... & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & \mu_r & 0 \end{pmatrix}, \tag{4.32}$$

where $\mu_i = i(r+1-i)$ for $1 \leq i \leq r$. The homomorphism for the nilpotent element labeled by d is

$$\Phi_d : sl_2 \to sl_n,\ via\ \Phi_d = \bigoplus_{1\leq i\leq k} \rho_{d_i-1}. \tag{4.33}$$

We can also find the commutator in SL_n of this homomorphism. Assume the nilpotent element associated with this homomorphism has the partition $d = [d_1, d_2, ..., d_k]$, let $r_i = |\{j|d_j = i\}|$, namely, r_i is the number of rows with parts i. The commutant is given by

$$G_{commu} = S(\prod_i (GL_{r_i})). \tag{4.34}$$

Using this homomorphism, we can construct the singular solution: replacing t_1, t_2, t_3 by $\Phi_d\{t_1, t_2, t_3\}$. The holomorphic part of the Higgs field has a simple pole at the singularity and the residue is the nilpotent element labeled by d. We want to associate these kind of solutions with four dimensional massless $\mathcal{N} = 2$ SCFT. The first discovery is: The singularity is labeled by partition of n for $\mathcal{N} = 2$ SU(n) SCFT.

We next study the behavior of spectral curve near the singularity to further confirm our conjecture. First consider the solution associated with the partition $[2, 1, ...1]$, We add regular terms to the solution so that the holomorphic part of the Higgs field is a regular semisimple element and looks like

$$\Phi(z)dz = \begin{pmatrix} * & (\frac{1}{z}+*) & * & ... & * & * \\ * & * & * & ... & * & * \\ . & . & . \ . & & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ * & * & * & ... & * & * \\ * & * & * & ... & * & * \end{pmatrix} dz + \mathcal{O}(z)dz, \tag{4.35}$$

where $*$ is the generic numbers so that this matrix is regular semisimple. We calculate

a) b)

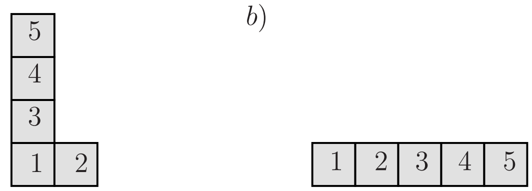


Fig. 9. Left: Young Tableaux with partition $[2,1,1,1]$, the order of poles are $p_1 = 1-1 = 0, p_2 = 2-1 = 1, p_3 = 3-2 = 1, p_4 = 4-3 = 1, p_5 = 5-4 = 1$. Right: Young Tableaux with partition $[5]$, the order of poles are $p_1 = 1-1 = 0, p_2 = 2-1 = 1, p_3 = 3-1 = 2, p_4 = 4-1 = 3, p_5 = 5-1 = 4$.

the determinant and expand it as a polynomial in x:

$$\det(x - \Phi(z)) = \sum_{i=2} (-1)^i p_i(z) x^{n-i}. \tag{4.36}$$

The coefficient p_1 is zero since the matrix is traceless, $p_i, i \geq 2$ has simple pole at $z = 0$ as we can see from calculating the determinant. Let's recall the rule of calculating the determinant: each term in determinant is derived by selecting numbers from the matrix, the rule is that there is only one item selected from one row and one column, we multiple those n selected terms.

For $p_l x^{n-l}$ term in our determinant, we select $n-l$ diagonal elements from the $\{(x-\Phi)_{22}...(x-\Phi)_{nn}\}$, we also select $\frac{1}{z} + *$ term from first row and a proper constant term from the second row of $(x - \phi(z))$. We see that the coefficient p_l is of order $\frac{1}{z}$. The result can be summarized from the corresponding Young Tableaux if we label the boxes as in Figure 9a), the pole of the coefficient is given by $p_i = i - s_i$, where s_i is the height of the ith box.

Next let's consider the solution labeled by the partition $[n]$, the matrix $\Omega =$

$(x-\Phi(z))dz$ (including the constant regular term)

$$(x-\Phi(z))dz = \begin{pmatrix} (x+*) &)(\frac{1}{z}+*) & * & ... & * & * \\ * & (x+*) & (\frac{1}{z}+*) & ... & * & * \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ * & * & * & ... & (x+*) & (\frac{1}{z}+*) \\ * & * & * & ... & * & (x+*) \end{pmatrix} dz + \mathcal{O}(z)dz. \tag{4.37}$$

Calculate the characteristic polynomial of this matrix and leave only the singular terms in z, we find that p_i has pole of order $(i-1)$. To show this, we simply expand the determinant and find the most singular term for the coefficient. For term $p_i x^{n-i}$, we select $(i-1)$ $(\frac{1}{z}+*)$ terms just above the diagonal terms, and then select the remaining $n-i$ diagonal terms, this is the maximal pole we can get at $z=0$. The order of pole can be read from the Young Tableaux, namely $p_i = i - s_i$, see Figure 9b).

For general partition, the matrix $(x-\Phi(z))$ has the form:

$$(x-\Phi(z))dz = \begin{pmatrix} I_{d_1} & * & * & ... & * \\ * & I_{d_2} & * & ... & * \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ * & * & * & ... & I_{d_k} \end{pmatrix} dz + \mathcal{O}(z)dz. \tag{4.38}$$

where I_{d_i} takes the form

$$I_{d_i} = \begin{pmatrix} x+* & (\frac{1}{z}+*) & * & ... & * \\ * & x+* & (\frac{1}{z}+*) & ... & * \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & (\frac{1}{z}+*) \\ * & * & * & ... & x+* \end{pmatrix} \tag{4.39}$$

The orders of pole for the coefficients $p_i, 2 \leq i \leq d_1$ are calculated as follows: we choose the diagonal terms from the other blocks except the first block I_{d_1}, then we do the same analysis on the first block as we do on the partition $[n]$; the order of pole is given by $i-1$ when $i \leq d_1$. To calculate term $p_{d_1+1}x^{n-d_1-1}$, We select $d_1 - 1$ terms of form $\frac{1}{z}$ and a constant term from first block I_{d_1}; We can not choose another $\frac{1}{z}$ term, since if we choose a $\frac{1}{z}$ term say coming from the first row from the second block, we can not choose the two diagonal terms adjacent to it in calculating the determinant, the maximal order of x we can get is $n - d_1 - 2$. Therefore, the order of pole is $d_1 - 1$, or $d_1 + 1 - 2$. The order of poles for other terms $p_i, d_1 < I \leq d_2$ is given by $i - 2$. We do the same analysis when we jump from d_i to d_{i+1}, in general, the order of pole is read from the Young Tableaux and given by $i - s_i$, where s_i is the height of the *ith* box.

2. Mass deformed theory

In this subsection, we are going to study what kind of singular solutions correspond to mass-deformed theory. Let's recall what we learned about SU(2) theory. The massless theory is associated with Higgs field whose residue is a nilpotent element Y_1 labeled by the partition $d = [2]$; what is actually important is the moduli space of solutions with appropriate boundary conditions so that the residue is living in the conjugacy class of Y_1. There is an isomorphism between this moduli space and nilpotent orbit itself. On the other hand, the mass deformed theory is described by a solution to Hitchin's equation so that the residue of the Higgs field is a semisimple element (which is also regular for $\mathfrak{su}(2)$). We also are concerned about the moduli space of solutions and there is also an isomorphism between the space of solutions and the semi-simple orbit itself. Both nilpotent orbit and semi-simple orbit are hyper-Kahler manifolds and the closure of nilpotent orbit is singular and semi-simple orbit can be thought of the deformation of nilpotent orbit. The basic requirement for this understanding is that they must have the same complex dimensions.

Generalizing above considerations of SU(2) to SU(N), we need to find certain kinds of solutions of Nahm's equation whose moduli space is a deformation of the moduli space of solutions we are studying in the last subsection. In general, given a triple (τ_1, τ_2, τ_3), let $\sigma_1, \sigma_2, \sigma_3$ be elements of g which commute with τ_j and satisfy the $\mathfrak{su}(2)$ relations, a solution to the equation is

$$a = \tau_1 + \frac{\sigma_1}{2s}, \quad b = \tau_2 + \frac{\sigma_2}{2s}, \quad a = \tau_1 + \frac{\sigma_3}{2s}, \quad s \to \infty. \tag{4.40}$$

These conditions mean that the residue of the Higgs field takes value in $\tau_2 + i\tau_3 + \sigma^c$, where σ^c is the nilpotent element we can get from $\mathfrak{su}(2)$ algebra $\sigma_1, \sigma_2, \sigma_3$. There is a one-to-one correspondence between the solution space with this boundary conditions and the adjoint orbit which contains $\tau_2 + i\tau_3 + \sigma^c$ (see appendix I for more details), it is also proved that this space is a hyper-Kahler manifold.

Since the nilpotent orbit is identified with the massless theory, we are led to think that the mass-deformed theory corresponds to semisimple orbit. The question is to identify the semi-simple orbit, we will call those semi-simple orbits as the mass-deformed orbits. The closure of the nilpotent orbit is singular and we can think of the mass-deformed orbit as the deformation of the nilpotent closure. The necessary condition for this is that the mass-deformed orbit has the same dimension as the closure of the nilpotent orbit.

The dimension for a nilpotent orbit is given by (4.28). The following lemma can be used to calculate dimension of a semi-simple orbit:

Let g be a reductive lie algebra and X is element in a semisimple orbit, its centralizer g^X is reductive and there exists a Cartan subalgebra h containing X. If Φ denotes the roots for the pair (g, h), then $g^X = h \bigoplus \sum_{\alpha \in \Phi_X} g_\alpha$, where $\phi_X = \{\alpha \in \phi | \alpha(X) = 0\}$.

We study sl_3 as an example to show how to use the above lemma to calculate the dimension of a semisimple orbit. The traceless diagonal matrices in sl_3, denoted as h, form a three dimensional Cartan subalgebra. For each $1 \leq i \leq 3$, define a linear functional in the dual space h^* by

$$e_i \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} = h_i. \tag{4.41}$$

The standard choices of positive and simple roots are

$$\Phi^+ = \{e_i - e_j | 1 \leq i < j \leq 3\} \; and \; \triangle = \{e_i - e_{i+1} | 1 \leq i \leq 2\}. \tag{4.42}$$

Consider the following matrices

$$X_1 = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & -(m_1 + m_2) \end{pmatrix}, \; X_2 = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & -2m_1 \end{pmatrix}, \; X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.43}$$

We now describe how to calculate dimension of semi-simple orbit $\mathcal{O}_{X_k}$ for $1 \leq k \leq 3$.

For case X_1, since $\alpha(X_1) \neq 0$ for any simple roots $\alpha \in \triangle$, g^{X^1} is a Cartan subalgebra using our lemma. The dimension for X_1 is

$$dim(\mathcal{O}_{X_1}) = dim(g) - dim(g^{X^1}) = 8 - 2 = 6. \tag{4.44}$$

For case X_2, $\alpha(X_2) = 0$ if and only if $\alpha = \pm(e_1 - e_2)$, so $g^{X_2} = h \bigoplus g_{e_1-e_2} \bigoplus g_{e_2-e_1}$ and $dim(\mathcal{O}_{X_2}) = 8 - 4 = 4$.

For case X_3, $\Phi_{X_3} = \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\}$, then $dim(g^{X_3}) = 8$, and $dim(\mathcal{O}_{X_3}) = 0$.

Let's study the semi-simple orbits for general sl_n algebra. We want to study semi-simple elements labeled by a partition $d = [d_1, d_2, ...d_k]$ of n. It has the form $X_d = diag(m_1, ..m_1, m_2...m_2, ...m_k..m_k)$, where the first d_1 diagonal terms have the same value, etc. It is interesting that we can also label semi-simple orbits by partitions of n. The dimension of the orbit $\mathcal{O}_{X_d}$ can be calculated by using the lemma we introduced above.

Let h be traceless diagonal $n \times n$ matrices; Define the linear functional $e_i \in h^*$ by $e_i(H) = i^{th}$ diagonal entry of H, here $1 \leq i \leq n$. The root system is $\{e_i - e_j | 1 \leq i, j \leq n, i \neq j\}$ in this representation. Elements in $\phi(X_d)$ from first block are

$$\Phi^1(X_d) = \{\pm(e_1 - e_2), \pm(e_1 - e_3), .\pm(e_1 - e_{d_1}), \pm(e_2 - e_3) \pm(e_2 - e_4)..\pm(e_2 - e_{d_1})...\pm(e_{d_1-1} - e_{d_1})\} \tag{4.45}$$

For the other block we can similarly find the other roots which satisfy the condition $\alpha(X_d) = 0$.

The dimension of the centralizer of X_d is

$$dim(g^{X_d}) = n - 1 + 2(d_1 - 1 + d_1 - 2 + ... + 1) + 2(d_2 - 1 + d_2 - 2 + .. + 1) + ... + (2d_k - 1 + ... + 1), \tag{4.46}$$

sum them up, we have

$$dim(g^{X_d}) = n - 1 + \sum_i^k d_i(d_i - 1) = n - 1 + \sum_i^k d_i^2 - n = \sum_i^k d_i^2 - 1. \qquad (4.47)$$

where the condition $\sum_i^k d_i = n$ is used. The dimension of the semisimple orbit is

$$dim(\mathcal{O}_{X_d}) = n^2 - 1 - \sum_i^k d_i^2 + 1 = n^2 - \sum_i^k d_i^2. \qquad (4.48)$$

Recall the dimension (4.28) of a nilpotent orbit with the partition d_1: $dim(\mathcal{O}_{X_{d_1}}) = n^2 - \sum_i s_i^2$, here s^i is the rows of the dual partition of d_1.

Comparing the dimension of the nilpotent orbits and that of the semi-simple orbits, the property of puncture with the partition d for the mass deformed theory is: It is described by the singular solution of Hitchin's equation and the Higgs field has simple pole at the singularity, whose residue is a semisimple element with the form labeled by the dual partition $d^t = [d_1^t, d_2^t, ...d_k^t]$ of d:

$$\Phi(z)dz = \frac{dz}{z} diag(m_1, ..m_1, m_2, ..m_2, ...m_k...m_k) + ..., \qquad (4.49)$$

where Φ has d_1 m_1 eigenvalues, d_2 m_2 eigenvalues and so on. The Seiberg- Witten curve is the spectral curve of the Hitchin's system.

The flavor symmetry can be read from the dual partition d^t directly. For this partition, a sl_2 homomorphism is defined and the commutant of this homomorphism in sl_n is given by

$$G_{commu} = S(\prod_i (GL_{r_i}), \qquad (4.50)$$

where r_i is the number of rows of d_t with boxes i. This number r_i is also the number of columns of d with heights i. The real form of this group is the flavor symmetry associated with the puncture and this agrees with Gaiotto's result. See [44] for the relevant discussion.

C. Generalized superconformal quiver gauge theory

In last section, we considered local behavior of Hitchin's equation near the regular punctures. Now we want to study some global properties.

In last chapter, we reconstructed known linear superconformal quiver gauge theory from six dimensional perspective and we found the local behavior of the singularity of the Hitchin equation from the known Seiberg-Witten curve. Surprisingly, these local behaviors are the only allowed behavior as derived from studying regular singular solutions from Hitchin equation.

Now nothing will prevent us from putting any number of punctures and any type of punctures on the Riemann surface. This will define a four dimensional $\mathcal{N} = 2$ theory. In brane's construction, the gauge theory is known first and we are trying to find its Seiberg-Witten curve. Here the situation is opposite, we know the Seiberg-Witten curve of a theory without knowing what the theory is. We also know the dimension of Coulomb branch which is calculated from dimension of Hitchin's moduli space with these specific boundary conditions at the puncture. The problem is to find what the gauge theory is. In fact, we would like to find its weakly coupled limit form.

Let's first discuss the UV deformations of gauge theory. The mass parameter is encoded in the local behavior of regular solution. The gauge couplings are realized as the complex structure of punctured Riemann surface. Then all the UV deformations are known. The question is to determine the gauge group and matter content. Of course, this depends on the duality frame. Happily, different weakly coupled duality frame is realized as the degeneration limits of the punctured Riemann surface.

1. The shape of generalized quiver from nodal curve

In the previous section, we argue that gauge coupling constants of four dimensional $\mathcal{N} = 2$ SCFT are identified as the complex structure of a Riemann surface with punctures. In this part, we show that we can determine the structure of quiver with weakly coupled gauge groups from studying the compactification of moduli space.

Consider a two dimensional topological surface Σ with g handles and n marked points. This manifold can be made into a complex manifold by defining a complex structure J on it. A complex structure J is a local linear map on the tangent bundle that satisfies $J^2 = -1$ and the integrability condition. Two complex structures are considered equivalent if they are related by a diffeomorphism. The moduli space $M_{g,n}$ is the space of all the inequivalent complex structure on the surface. By Riemann-Roch this is a space of complex dimension

$$dim M_{g,n} = 3g - 3 + n. \tag{4.51}$$

$M_{g,n}$ is a noncompact complex space with singularities. It arises as the quotient of a covering space known as Teichmuller space $T_{g,n}$, by a discrete group, conformal mapping class group $MC_{g,n}$:

$$M_{g,n} = \frac{T_{g,n}}{MC_{g,n}}. \tag{4.52}$$

This action typically has fixed points, and the moduli space has orbifold singularities.

There is another useful way to think about the complex structure on Σ. We can think of the point on the moduli space as the conformal class of a metric $g_{\mu\nu}$. Indeed, a metric defined a complex structure through

$$J_\mu{}^\nu = \sqrt{h}\epsilon_{\mu\lambda}h^{\lambda\nu}, \tag{4.53}$$

with $\epsilon_{\mu\nu}$ the Levi-Civita symbol. The definition of the complex structure does not

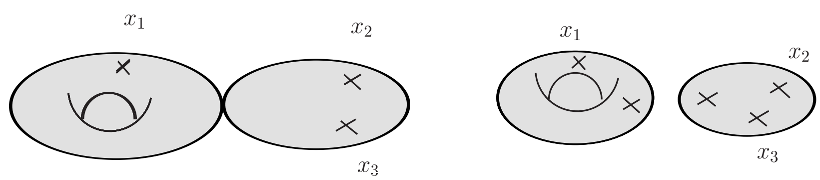


Fig. 10. Left: A nodal curve. Right: The normalization of a.

depend on the local resealing of the metric $g_{\mu\nu}$, so we can think of the moduli space as the spaces of metric modulo local rescalings and diffeomorphisms.

The moduli space $M_{g,n}$ is noncompact and has a boundary. The boundary points can be intuitively represented as degenerate surfaces. The degeneration can be thought in two ways; the surface can either form a node-or equivalently a long neck- or two marked points can collide. The process in which two points x_1 and x_2 collide if $q = x_1 - x_2$ tends to zero can alternatively be described as a process in which a sphere, that contains x_1 and x_2 at fixed distance, pinches off the surface by forming a neck of length $\log q$. So the degeneration limit can be thought of the nodal curve. The boundary points can be thought of as in the infinity and we would like to compactify this space. The Deligne-Mumford compactification of $M_{g,n}$ is achieved by adding some points which represent stable nodal curves.

In the following, we will introduce some basic concepts about the nodal curve. Singular objects play an important role in algebraic geometry. The simplest singularity a complex curve can have is a node. A nodal point of a curve is a point that can be described locally by the equation $xy = 0$ in C^2. An example is shown in Figure 10a).

We also find the following description of nodal curve very useful. On a surface with node, the node separates the surface into two components, on the neighborhood of each node, we can choose local coordinate disks $\{z_i : |z_i| < 1\}, i = 1, 2$. The two disks are glued together at the origin $z_1, z_2 = 0$ to form the node. We can open the node by introducing one of complex coordinate q of the moduli space $M_{g,n}$. Remove the sub-disks $|z_i| < |q|^{\frac{1}{2}}$ and attach the resulting pair of annuli at their inner boundaries $|z_i| = |q|^{\frac{1}{2}}$ by identifying $z_2 = q/z_1$. This coordinate neighborhood on the surface is mapped to a single annulus $|q|^{\frac{1}{2}} < |z| < |q|^{\frac{-1}{2}}$, by

$$z = q^{1/2}/z_2, \quad if \; |q|^{1/2} < |z| \leq 1,$$
$$z = q^{1/2}/z_2, \quad if \; 1 \leq |z| < |q|^{-1/2}. \tag{4.54}$$

As $q = 0$, we recover the node. A further transformation $\omega = (2\pi i)^{-1} lnz$ pictures the opened node as a long tube. Writing $q = e^{2\pi i \tau}$, the length and width is determined by τ. The node corresponds to a tube of infinite length. In this description, we see that the moduli is localized on the long tube, and since we identify the moduli with the gauge coupling constant, we can think that the gauge group is represented by the long tube.

We define the normalization of the nodal curve as unglueing its nodes, and add a marked points to each of the components on which the nodes belong to. See Figure 10b) for an example. Each component Σ_i after the normalization is an irreducible component of Σ.

There is another convenient way of describing the nodal curve by drawing a dual graph. The vertices of the dual graph of Σ corresponds to components of Σ (and are labeled by their genus), and the edge correspond to node, we use labeled tails to represent the marked points. An example is shown in Figure 11.

A stable nodal curve is a connected nodal curve such that:

(i) Every irreducible component of geometric genus 0 has at least three special

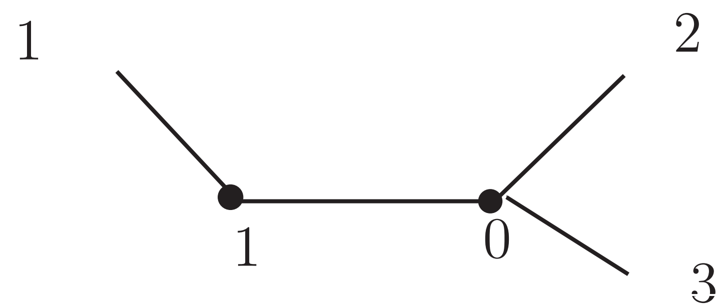


Fig. 11. A dual graph for the nodal curve in Figure 10.

points (including the marked points and the nodal points after the normalization).

ii) Every irreducible component of geometric genus 1 has at least one special point.

Deligne-Mumford compactification $\bar{M}_{g,n}$ includes the points corresponding to the stable curve to moduli space $M_{g,n}$.

Let's define an irreducible nodal curve as a curve whose irreducible components are all genus 0 curve with three special points. See Figure 12 for an example, The dual graph for this particular nodal curve is depicted in Figure 13a.

Let's consider another genus one example, in this case, the two nodes belong to the same irreducible component after normalization. The degeneration limit and the dual graph are shown in Figure 14.

It is time now to connect the nodal curve to the weakly coupled four dimensional $\mathcal{N} = 2$ quiver we are studying in last section. As we reviewed in last section, each puncture is associated with certain flavor symmetry, and the node or the long neck is identified with the weakly coupled gauge group, we have the following identification:

A generalized quiver with weakly coupled gauge group associates with the stable nodal curve and the quiver with all gauge group weakly coupled is the irreducible

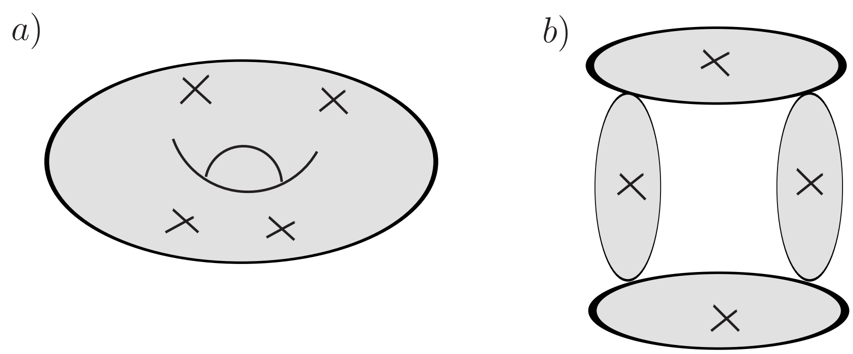


Fig. 12. Left: A torus with four marked points. Right: An irreducible nodal curve of a.

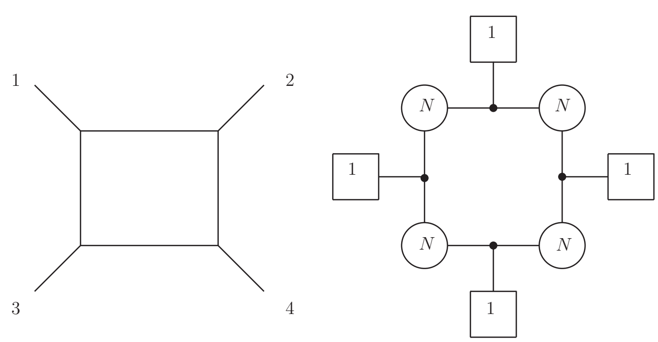


Fig. 13. Left: The dual graph for the irreducible nodal curve of Figure 12, we omit the genus 0 on each vertex for simplicity, since for irreducible nodal curve, all components have genus zero. Right: Four dimensional gauge theory, we put a gauge group on the internal line, external lines represent the U(1) flavor symmetries.

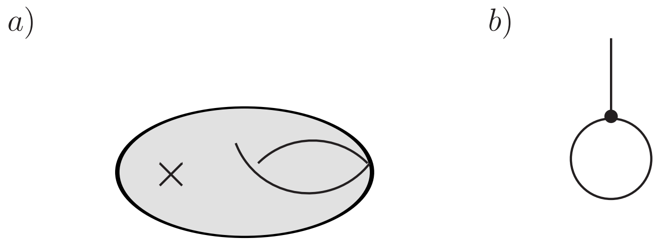


Fig. 14. Left: Degeneration limit of torus with one marked point. Right: The dual graph of a), there is only one irreducible component and the nodes are in the same component so the nodes is represented by a loop connecting to the same vertex.

nodal curve.

In fact, we can read the quiver structure from the dual graph in Figure 13a; Let's assume that we compactify A_{N-1} theory on the Riemann surface, and the punctures are simple punctures. We consider the weakly coupled gauge theory corresponding to irreducible nodal curve. The line ending on only one vertex is the original puncture and represent the flavor symmetry, we call them external line; The line between the nodes represent the gauge groups, we call them internal line. There are three lines connecting each node. In this particular example, for each node, there are two internal lines connecting it and one external line representing a $U(1)$ flavor symmetry, the gauge theory interpretation is that the two gauge groups connecting to a single node are adjacent and there are bi-fundamental fields connecting them, and the quiver is of the form in Figure 13b. In general, we can read the shape of the generalized quiver in any duality frame from the dual graph of the corresponding nodal curve, the difference is that the matter fields between the adjacent gauge group is not necessarily bi-fundamental fields, it maybe a strongly coupled isolated superconformal field theory like E_6 SCFT.

Since for an irreducible nodal curve, each irreducible component is a genus 0 Riemann sphere with three punctures, we may think that each three punctured represents a matter either conventional bi-fundamental fields or strongly coupled isolated SCFT matter. The whole generalized quiver is derived by gluing the matter fields. The gluing process is to gauge the diagonal flavor symmetry of two punctures on two different irreducible components or one component, in the latter case, we add a handle to the Riemann surface. This is true for generalized quiver gauge theory defined by six dimensional A_1 theory with any number of punctures; For A_N theory, in certain duality frame (for certain irreducible nodal curve) this is true, but generically this is not the case, since some of the three punctured sphere does not represent a matter, we will discuss this more in later sections.

Let's summarize what we learn about the relation between the stable nodal curve and the four dimensional $\mathcal{N} = 2$ weakly coupled quiver gauge theory. For each stable nodal curve, there is a four dimensional gauge theory for which one or more than one gauge groups become weakly coupled, and the gauge couplings are taking value at the boundary of the moduli space $M_{g,n}$. The four dimensional quiver for which all the gauge groups are weakly coupled corresponds to the irreducible nodal curve. In another words, the four dimensional $\mathcal{N} = 2$ SCFT gauge coupling space is $\bar{M}_{g,n}$.

It is illuminating to note that $\bar{M}_{g,n}$ also plays an important role in 2d conformal field theory. Let's consider a two dimensional conformal field theory defined on a Riemann surface. We usually want to calculate the correlation functions with several insertions on the Riemann surface, and the correlation function can be calculated in different channels, say s, t channels. Recently, AGT [19] found an interesting relation between the partition function of $\mathcal{N} = 2$ SU(2) four dimensional gauge theory and the correlation function of the Liouville theory based on the six dimensional realization. The primary fields at the insertion can be read from the information of the puncture

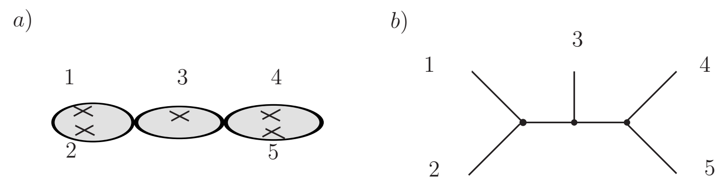


Fig. 15. Left: A nodal curve with 5 marked points and two vertices. Right: The dual graph which resembles the conformal field theory conformal block with 5 external states.

used to describe the gauge theory. It is interesting to note that the different channels for two dimensional correlation function are also in one-to-one correspondence with the nodal curve. The different channels are exactly represented by the dual graph of the nodal curve, the external states are represented as the external line while the intermediate states are represented by the internal line . The intermediate states can be determined by using familiar operator product expansion technics based on the fixed external states. See Figure 15 for an example.

Using the nodal curve, we now establish a one-to-one correspondence between four dimensional gauge theory and two dimensional CFT correlation function. The external lines on dual graph determine the flavor symmetry of the gauge theory and the external states of the two dimensional correlation function, the classification of punctures is given in last section and this leads to the determination of physical states in 2d CFT. The internal lines represent gauge group of gauge theory and the intermediate state of the 2d CFT. We need to determine the gauge group and then we can identify the intermediate state in 2d CFT.

2. The gauge group

In last subsection, we established a relation between $\mathcal{N}=2$ weakly coupled SCFT and the nodal curve and the shape of the quiver is determined. The remaining task is to determine what is the weakly coupled gauge group or find the representation of the quiver. There is a Hithcin's system defined on the punctured Riemann surface whose moduli space is identified with the Seiberg-Witten fibration. The decoupled gauge group is by taking the complete degeneration limit of the Riemann surface and comparing the dimension of Hitchin's moduli space.

Let's first consider degeneration limit of punctured Riemann sphere. After degeneration, the original Riemann surface decomposes into two punctured Riemann spheres. From gauge theory point of view, one of the gauge group is decoupled, and there are two subquivers are left (sometimes there is no clear meaning for one or two components). We assume that the decoupled gauge group is a simple gauge group with the form $SU(k),\ k\leq N$ or $USp(2k),\ k\leq[\frac{N}{2}]$. This assumption is confirmed in all known example. The form of the decoupled gauge group is derived by matching the Coulomb branch moduli after decoupling with the original quiver.

It is useful to define irreducible rank N theory on punctured Riemann sphere. The Seiberg-Witten curve for theory from punctured sphere is

$$x^N+\phi_i x^{N-i}=0. \tag{4.55}$$

Here $\phi_i dz^i$ is a degree i meromorphic differential defined on Riemann sphere, the dimension d_i of this differential is

$$d_i=\sum_j p_i^{(j)}-2i+1, \tag{4.56}$$

here p_i^j is the order of pole at the jth puncture. $p_i^{(j)}$ can be read from the Young

Tableaux:

$$p_i^{(j)} = i - s_i, \tag{4.57}$$

where s_i is the height of ith box in the Young Tableaux. Consider the degree N differential, if the number $d_N \leq 0$, then the Seiberg-Witten curve degenerates as (we consider massless theory here)

$$x(x^{N-1} - \phi_i x^{N-1-i}) = 0, \tag{4.58}$$

so actually this theory can be realized as a rank $(N-1)$ theory if $d_{N-1} > 0$. We call a theory defined by A_{N-1} compactified on a punctured Riemann surface irreducible if $d_N > 0$.

In fact, only the case with two punctures colliding is needed to consider. We assume that the new appearing puncture has the same local behavior as other singularities. Then we can consider collision of this new appearing puncture with another puncture, in this way, the decoupled gauge group can be found when any number of punctures are colliding.

Let's consider an irreducible rank N theory derived from a Riemann sphere with n punctures. When two punctures are colliding, we are left with a three punctured sphere called part 1 and a $n-1$ punctured part called part 2. The new punctures on two parts are the same. An important relation between the new puncture and the decoupled gauge group is that: the decoupled gauge group is a subgroup of the flavor group associated with the new puncture. Physically, this means the original gauge theory is formed by gauging the subgroup of the new puncture. There might be some more fundamentals decoupled too.

It is enough to specify the Young Tableaux associated with the new puncture. This is equivalent to determine the order of pole of degree i differential at this new puncture. This is achieved by matching the number of moduli with the original quiver. Consider the degree i moduli, assume the two colliding punctures contribute to $\delta_{1i} = p_i^{(1)} + p_i^{(2)}$, and the other $n-2$ punctures contribute to δ_{2i}. Let's first assume that both components have zero or non-zero degree i moduli, this puts the constraint on δ_{1i} and δ_{2i}

$$(1) : \delta_{1i} \geq i \quad and \quad \delta_{2i} \geq i. \tag{4.59}$$

There are two options to consider. First if the decoupled gauge group does not have a degree i operator, then we have

$$(\delta_{1i} + p_i - 2i + 1) + (\delta_{2i} + p_i - 2i + 1) = \delta_{1i} + \delta_{2i} - 2i + 1, \tag{4.60}$$

where p_i is the contribution from the new puncture to the ith degree moduli. The left is the contribution of part 1 and part 2 to degree i moduli. On the other hand, if the decoupled gauge group carry just one degree i moduli (this is the only choice by our assumption of the decoupled gauge group)

$$(\delta_{1i} + p_i - 2i + 1) + (\delta_{2i} + p_i - 2i + 1) + 1 = \delta_{1i} + \delta_{2i} - 2i + 1. \tag{4.61}$$

The first option gives $2p_i - 2i - 1 = 0$ which is inconsistent since p_i and i are both integer. For the second option, we have $p_i = i - 1$. So we conclude that $p_i = i - 1$ with constraint (1).

Next let's consider only one component has degree i moduli, this implies

$$(2) : \delta_{1i} \geq i \quad and \quad \delta_{2i} < i; \;\; or \;\; (3) : \delta_{1i} < i \quad and \quad \delta_{2i} \geq i. \tag{4.62}$$

For case (2), part 2 does not have a degree i operator since the maximal contribution of the new appearing puncture to degree i differential is $p_i = i - 1$, and the maximal number of degree i operator is $d_i^{(2)} \leq \delta_{2i} + (i-1) - 2i + 1 < 0$. There are also two options, if decoupled gauge group does not carry a degree i operator, we have

$$(\delta_{1i} + p_i - 2i + 1) = \delta_{1i} + \delta_{2i} - 2i + 1. \tag{4.63}$$

If the decoupled gauge group has a degree i operator, the equation is

$$(\delta_{1i} + p_i - 2i + 1) + 1 = \delta_{1i} + \delta_{2i} - 2i + 1, \tag{4.64}$$

Here p_i is the contribution to the degree i operators of the new appearing puncture. Solving the equations, the solution is $p_i = \delta_{2i}$ or $p_i = \delta_{2i} - 1$. The same analysis can be applied to case (3), we have $p_i = \delta_{1i}$ in the first case and $p_i = \delta_{1i} - 1$ in the second option. The second option is not possible by further analysis on the relation of the decoupled gauge group and the flavor group of the new puncture.

Let's just consider the case (3). Since $\delta_{1i} \leq i$, write $\delta_{1i} = i - a$ with $a \geq 1$. If $p_i = \delta_{1i} - 1 = i - (a+1)$, then the ith box is at the level $(a+1) \geq 2$ in the Young Tableaux of the new puncture. Since the decoupled gauge group has a degree i operator, so the decoupled gauge group is at least $SU(i)$ or $USp(i)$ ($USp(i)$ is possible with even i). However, for the new puncture $n_1 < i$ since ith box is not in the first row, the maximal simple subgroup of the flavor symmetry is less than $SU(i)$. Therefore, the decoupled gauge group is large than the flavor group of the new puncture which contradicts our assumption.

The case $\delta_{1i} < i$ and $\delta_{2i} < i$ is excluded since this implies the original quiver has negative number of degree i operator.

Combining all the analysis above, we can give a concise formula for p_i

$$p_i = min(\delta_{1i}, \delta_{2i}, i - 1). \tag{4.65}$$

and if $min(\delta_{1i}, \delta_{2i}) \geq i$, there is a degree i operator for the decoupled gauge group. Indeed, in the derivation of the above formula, the assumption that only two punctures are colliding is not used, so this formula is true for colliding any number of punctures. To determine the fundamental fields on the decoupled gauge group, we need to match the dimension of Higgs branch. This can be done in a similar way as the Coulomb branch.

We next consider the degeneration limit of higher genus theory. Let's study Riemann surface with genus g and n marked points; there are now three kinds of degeneration: the genus reduces by one, or two marked points collide and there are a genus g component and a genus zero component left; Finally there are a genus g_1 and genus g_2 components with g_1 and g_2 are nonzero.

In the first case, there is only a genus $g-1$ surface with $n+2$ marked points left. Denote the local dimension of the new puncture as d, we have

$$\frac{1}{2}\sum d_i + \frac{1}{2}(2d) + (g-1-1)(N^2-1) + r = \frac{1}{2}\sum d_i + (g-1)(N^2-1), \quad (4.66)$$

where r is the rank of the decoupled gauge group and d_i is the dimension of nilpotent orbit associated with the puncture i, (The total dimension of Hitchin's moduli space on a genus g Riemann surface is $\sum d_i + 2(g-1)(N^2-1)$, half of this number is the dimension of the Coulomb branch). Solving the above equation, we have

$$d = N^2 - (r+1). \quad (4.67)$$

the maximal dimension of d is the dimension of regular nilpotent orbit and has the dimension $d = N^2 - N$, this implies that the minimal value of r is $N-1$. However, the maximal rank of the decoupled gauge group is $(N-1)$. We conclude that the decoupled gauge group is SU(N) and the new puncture is a full puncture. The original theory is assumed to be irreducible and we can check genus $(g-1)$ theory is also irreducible. The irreducibility of the original theory implies

$$d_g = \frac{1}{2}\sum_i d_i + (g-1)(N^2-1) > 0, \tag{4.68}$$

This condition is automatically good if $g \geq 2$, there is no constraint on the number of punctures. In the case $g = 1$, we need to have at least one puncture.

For the genus $g-1$ theory, we have the dimension of the Coulomb branch

$$d_{g-1} = \frac{1}{2}\sum_i d_i + N^2 - N + (g-2)(N^2-1) = \frac{1}{2}\sum_i d_i + (g-1)(N^2-1) - (N-1) \tag{4.69}$$

In the case $g > 2$, $d_{g-1} > 0$ is always true. In the case of $g = 1$, since the minimal dimension of the nilpotent orbit is $2N-2$, we see that $d_{g-1} \geq 0$. This result shows that the handle of the Riemann surface can only be formed by a SU(N) group.

The result can be confirmed by matching Higgs branch moduli using (5.18). The matching condition is

$$\sum_i l_i + 2l + (1-(g-1))(N-1) - n = \sum_i l_i + (1-g)(N-1), \tag{4.70}$$

where l is the contribution of the new puncture and n is the dimension of the decoupled gauge group, which is $n = (N^2-1)$ in our case. The new puncture is a full puncture and have $l = \frac{1}{2}(N^2-N)$. One can check the above equation is right. This calculation also shows that we do not have any fundamental fields on the SU(N) gauge group.

The degeneration limit with genus g_1 and g_2 parts can be analyzed similarly. The g_1 component has n_1+1 marked points and g_2 component has n_2+1 marked points, according to our previous analysis, these two theories are both irreducible. We have the following relation for the coulomb branch dimension

$$\begin{aligned}\sum k_{1i} + \frac{1}{2}d + (g_1-1)(N^2-1) + \sum k_{2i} + \frac{1}{2}d + (g_2-1)(N^2-1)\\ = \sum (k_{1i}+k_{2i}) + (g_1-g_2-1)(N^2-1) - r.\end{aligned} \tag{4.71}$$

where r is the rank of the decoupled gauge group and d is the dimension of the nilpotent orbit associated with the puncture as we defined above. Similar analysis shows that the decoupled gauge group is SU(N) and the new puncture is a full puncture.

The last case with a genus g component and genus zero component is a little bit different. We know that a genus g component is irreducible, there are nonzero moduli for each degree. Assume the contribution of two punctures to the moduli of degree i is δ_{1i}, similar analysis with the degeneration limit of genus zero case can be done and we have the following conclusion about the order of poles of the new puncture

$$p_i = min(\delta_{1i}, i-1). \tag{4.72}$$

The decoupled gauge group can be derived by noticing that if $\delta_{1i} \geq i$, the decoupled gauge group has a degree i operator.

Now let's discuss what is the intermediate state in A_{N-1} conformal Toda field theory side. It is argued in [45], that the primary field corresponding to the simple puncture labeled by the Young Tableaux $[n_1, n_2, n_3, ...n_s]$ has the form

$$e^{i\vec{\beta}.\vec{\phi}}, \tag{4.73}$$

where $\overrightarrow{\phi} = (\phi_1, ...\phi_N)$ and $\sum \phi_i = 0$, and β has the form

$$\vec{\beta} = \vec{p} - iQ\vec{\rho}, \tag{4.74}$$

ρ is a fixed vector and $\vec{p}$ is a real vector, they are both dictated by the Young Tableaux. $\vec{p}$ has the form:

$$\vec{p} = (\underbrace{p_1, ...p_1}_{l_1}, \underbrace{p_2, ...p_2}_{l_2} .... \underbrace{p_r, ...p_r}_{l_r}), \tag{4.75}$$

here l_i is the height of ith column of the Young Tableaux. Notice that in the gauge theory, the mass deformation at the puncture has the same form as (4.75), so we identify the mass parameters with the momentum $\vec{p}$ and these numbers are fixed (they are UV parameters).

The intermediate state also has the form (4.75) and the Young Tableaux of it is dictated by the new appearing puncture. The physical momentum of the intermediate state is identified with the Coulomb branch expectation value. For instance, if the new appearing puncture is the full puncture and the decoupled gauge group is SU(N), then the intermediate state has the form $\vec{p} = (a_1, a_2, ...a_N)$, $\sum a_i = 0$, here $a_i, i \geq 2$ parameterizes the Coulomb branch of $SU(N)$. However, in general, not every column of the new appearing puncture can be deformed since not all flavor symmetry of it is gauged. The practical rule is: if $min(\delta_{1i}, \delta_{2i}) \geq i$, the decoupled gauge group has a degree i operator; since $p_i = i - 1$ and the ith box of the new Young Tableaux is at the first row, we claim that the ith column of the new puncture is deformed (we always deform first column so that the traceless condition is satisfied).

3. Examples

We apply our formula to calculate the decoupled gauge group for some examples in this subsection. The six dimensional description of a linear quiver of A_N type has been worked out by Gaiotto [11], it involves two generic puncture, and several basic punctures.

We decide what happens when a simple puncture is colliding with a generic puncture with rows $n_1 \geq n_2 ... \geq n_k$, and the height of the first column is s_1. We assume part 2 is an irreducible rank N theory. Since part2 has a gauge group SU(N) and has degree i moduli, $\delta_{2i} \geq i$. Then the new appearing puncture is determined by $p_i = \min(\delta_{1i}, i - 1)$. The numbers is in the Table I. In the last line of Table I, we indicate whether decoupled gauge group has a degree i operator. We can see

i	2	3	4	...	n_1	n_1+1	...	N
p_{1i}	1	1	1	...	1	1	...	1
p_{2i}	1	2	3	...	n_1-1	n_1-1	...	$N-s_1$
δ_{1i}	2	3	4	...	n_1	n_1	...	$N-s_1+1$
p_i	1	2	3	...	n_1-1	n_1	...	$N-(s_1-1)$
	1	1	1	...	1	0	...	0

Table I. The data needed for colliding a generic puncture and a simple puncture.

from the Table I that the decoupled gauge group is $SU(n_1)$, and the new puncture has the feature that the first row and second row are combined and other rows are unchanged. This agrees with the result by Gaiotto. We can now determine the fundamentals on the gauge group $SU(n_1)$, since the three punctured sphere does not carry any Coulomb branch moduli and the dimension is negative, there is no contribution to Higgs branch from it. We have

$$\sum_{i=2}^{n} l_i + l_2 + 1 + (N-1) = \sum_{i=2}^{n} l_i + l_2 + n_1 n_2 - (n_1^2 - 1) + (N-1) + x, \tag{4.76}$$

Here l_2 is the contribution of the generic puncture, 1 is the contribution of the simple puncture; we have used the fact that the new puncture has the contribution to Higgs branch $(l_2+n_1 n_2)$, where x is the contribution from the fundamental fields. Calculate it, we get $x = n_1(n_1 - n_2)$, so there is $n_1 - n_2$ fundamentals on $SU(n_1)$, this is in agreement with the explicit quiver theory.

Next, let's consider collision of two identical punctures which have two columns with equal height N, this means that we are considering A_{2N-1} compactification. We list the analysis in the Table II: From the Table II, we can see that the new puncture is a full puncture and the decoupled gauge group has only even rank operator,

i	2	3	4	...	$2k$	$2k+1$	...	2N
p_{1i}	1	1	2	...	k	k	...	N
p_{2i}	1	1	2	...	k	k	...	N
δ_{1i}	2	2	4	...	$2k$	$2k$	...	$2N$
p_i	1	2	3	...	$2k-1$	$2k$	...	$2N-1$
	1	0	1	...	1	0	...	1

Table II. The data needed for collision of two identical punctures with equal height N.

the natural decoupled gauge group is $USp(2N)$, one may wonder why USp gauge group appears when we compactify a A_{2N-1} theory on a Riemann surface, this can be done by including a outer automorphism of the gauge group $SU(2N)$ in the compactification, see [46, 47]. Let's calculate the fundamentals on $USp(2N)$. The three punctured sphere does not contribute to Higgs branch. The two column puncture has Higgs dimension N, the full puncture contributes $(2N^2-N)$. we have

$$\sum_{i=2}^{n} l_i + 2N + (2N-1) = \sum_{i=2}^{n} l_i + (2N^2-N) + (2N-1) - 2N^2 - N + x, \quad (4.77)$$

solving this equation, we have $x = 4N$, so we have 2 fundamentals on USp node. This is just what is found by Gaiotto [11], see also [21].

We also confirm another example which is studied in [21]. One puncture has partition $[2, 2...2]$, the other puncture has partition $[3, 2....2, 1]$, the data is assembled in Table III: From the Table III, we conclude that the new puncture is a full puncture and the decoupled gauge group is a $SU(2N)$ gauge group. The three punctured sphere does not contribute to higgs branch, we have the equation

$$\sum_{i=2}^{n} l_i + N + (N+1) + (2N-1) = \sum_{i=2}^{n} l_i + (2N^2-N) + (2N-1) - ((2N)^2-1) + x, \quad (4.78)$$

i	2	3	4	...	$2k$	$2k+1$	...	2N
p_{1i}	1	1	2	...	k	k	...	N
p_{2i}	1	2	2	...	k	$k+1$	...	N
δ_{1i}	2	3	4	...	$2k$	$2k+1$	...	$2N$
p_i	1	2	3	...	$2k-1$	$2k$	...	$2N-1$
	1	1	1	...	1	1	...	1

Table III. The data needed for colliding two punctures appearing in $SU(N)$ theory with antisymmetric matter.

we find $x = 2N^2 + N$, this sounds weird, since no single conventional matter on SU(N) can give this number for Higgs branch. However, let's split $2N^2 + 3N = (2N^2 - N) + 4N$, that's an antisymmetric matter and two fundamentals' contribution, it splits in this way so that the SU(N) gauge group is conformal.

Next, let's consider the collision of two generic punctures. First we want to mention a special case when $\delta_{1N} \geq N - 1$, this means that $p_N = N - 1$, without any calculation, we can conclude that the new puncture is a full puncture, since the maximal value of p_N is $N - 1$ and it is possible only if the puncture is a full puncture.

Finally, let's consider an example of collision of two generic punctures, in some cases, the new appearing three punctured sphere has nonzero moduli and is an isolated SCFT. Lets's consider collision of two identical punctures with partitions $[3, 1, 1, 1]$. The linear quiver gauge theory with these two punctures is depicted in Figure 16a. The six dimensional construction is depicted in Figure 16b. We study another weakly coupled theory corresponding to collide two generic puncture represented by black dot, the nodal curve is depicted in Figure 16c. We write the corresponding generalized quiver corresponding in Figure 16d.

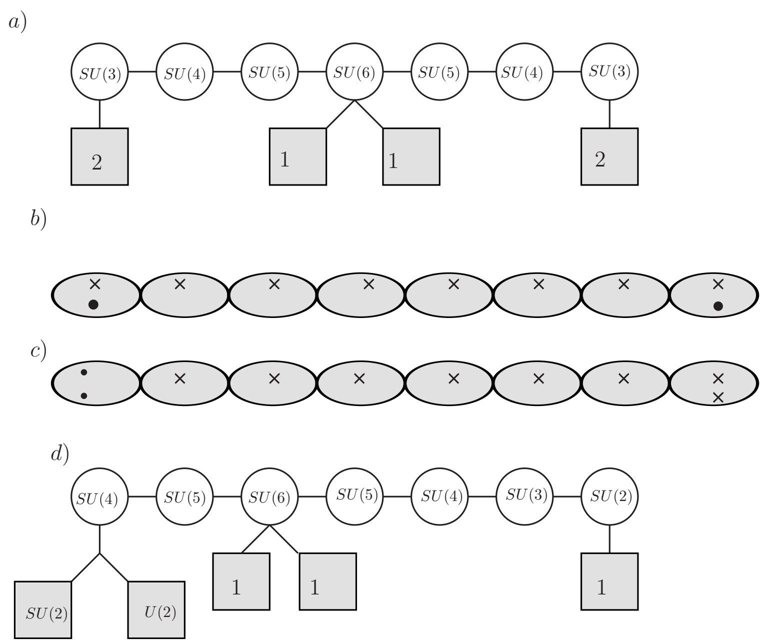


Fig. 16. a): A linear quiver. b): The six dimensional construction corresponding to quiver in (a), the cross denotes the simple puncture and the black dot denotes the puncture with partition $[3, 1, 1, 1]$. c): A different weakly coupled gauge group description, we collide two generic punctures. d): A generalized quiver corresponding to (c).

i	2	3	4	5	6
p_{1i}	1	2	2	2	2
p_{2i}	1	2	2	2	2
δ_{1i}	2	4	4	4	4
p_i	1	2	3	4	4
	1	1	1	0	0

Table IV. The data needed for colliding two generic punctures.

We do need to know what happened when we collide two generic punctures, we follow our method and the data we need is in Table IV.

The new appearing puncture has the partition $[5, 1]$, the decoupled gauge group is SU(4). The decoupled three punctured sphere is reducible but it carries a degree 3 moduli as we can check explicitly. To see what this theory is, we reduce the three punctured sphere to a rank 3 theory, then all the three puncture now is the full puncture (we simple delete the extra boxes for each puncture), this theory is the familiar E_6 SCFT. Notice in this case, we use the subgroup $\mathrm{SU}(2) \times \mathrm{U}(2) \times \mathrm{SU}(4)$ decomposition of E_6 instead of the familiar $\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$ decomposition. After gauging the SU(4) node, we are left a $\mathrm{SU}(2) \times \mathrm{U}(2)$ flavor symmetry. Combining the U(1) flavor symmetry on SU(6) node, we have $\mathrm{U}(2) \times \mathrm{U}(2)$ flavor symmetry on this quiver tail, and these are represented by two generic punctures with partition $[3, 1, 1, 1]$. To confirm our identification of E_6 theory, we can match the Higgs branch dimension. According to our rule, the contribution of three punctured sphere should be calculated using rank 3 theory, and its Higgs branch has dimension 11. The total dimension of Higgs branch of quiver depicted in Figure 16a) is 19 using our formula (5.18). The Higgs branch dimension of the quiver in Figure 16d) after the complete degeneration is

$$23 + 11 + x - 15 = 19, \tag{4.79}$$

where 23 is from the left quiver and x is the contribution from the fundamental fields, we have $x = 0$. This result shows that there is no extra fundamentals on SU(4) node, which means that E_6 matter system provides conformal anomaly like the three fundamentals on SU(4) node, one can check this using the method in [10].

4. Three punctured sphere and sewing SCFT

The punctured Riemann surface can be derived by gluing the three punctured spheres. In gauge theory terms, we may think that each three puncture corresponds to a matter system, say a bi-fundamental field or a strongly coupled SCFT like E_6 theory [48], and the whole generalized quiver gauge theory is derived by gauging various flavor symmetry of the matter system represented by the three punctured sphere. This is obvious true since the as we saw the three punctured sphere after degeneration has negative coulomb branch parameter, so these three punctured spheres can not be seen as the matter system.

In some cases, all the three punctured spheres represent the matter system. This is true for any generalized SU(2) quiver gauge theory, since this theory has only one type puncture and one type three punctured sphere, which represents the bi-fundamental fields with flavor symmetry $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, any generalized SU(2) quiver gauge theory in any duality frame is derived by gluing the flavor symmetry of the basic three punctured sphere. Another interesting example is $\underbrace{\mathrm{SU}(N) - \mathrm{SU}(N) - ... - \mathrm{SU}(N)}_{n}$ gauge theory with N fundamentals at each end, this theory is described as six dimensional A_{N-1} theory compactified on a sphere with $n+1$ simple punctures and two full punctures. At one duality frame corresponding to the conventional description, the six dimensional nodal curve consists of three punctured sphere with two full punctures and one simple puncture, each three punctured sphere represents the bi-fundamental fields between two SU(N) gauge groups. However, this same statement is not true in other duality frames. Another example is the theory derived by compactifying six dimensional theory on a genus g Riemann surface, as we discussed in previous subsections, in the complete degeneration limit, all three punctured sphere has three full punctures which is the so-called T_N theory, see [11, 49]. This fact is true in any duality frame.

A three punctured sphere represents a real matter system if the three punctured sphere is irreducible or the moduli space is just a point. The E_6 SCFT theory is irreducible while the bi-fundamental fields between two SU(N) gauge group has zero-dimensional moduli space.

Let's now discuss the interesting relation with conformal field theory. For Liouville theory, the correlation function is decomposed as the product of the three point functions and the conformal block [50]. This has the nice correspondence in gauge theory since each three punctured sphere in any duality frame represents a real matter system.

In the conformal Toda theory, it is shown that generically the correlation function can not be written as the product of three point function and the conformal block [51, 52, 53], however, in some special cases the correlation function is factorizable. There is one example shown in [51], the gauge theory is $\underbrace{SU(N) - SU(N) - ... - SU(N)}_{n}$ we considered previously, and all the three punctured spheres in this weakly coupled duality frame represent real matter system. Notice that in other duality frame, some of the three punctured sphere does not represent a real matter system and the correlation function in that channel is not factorizable. We then have this conjecture:

The correlation function in one channel of two dimensional CFT is factorizable if all the three punctured sphere for the corresponding weakly coupled gauge description represents real matter system.

Since each channel of 2d CFT correlation function corresponds to a duality frame of the gauge theory, this observation makes a completely determination about whether the correlation function can be written as the product of the three point functions and the conformal block.

CHAPTER V

ASYMPTOTICAL FREE THEORY, ARGYRES-DOUGLAS POINT: IRREGULAR SINGULARITY

The above analysis are all based on four dimensional superconformal field theories. It is interesting to extend the six dimensional construction to asymptotical free theories. It seems that in this case the irregular singularity of the Hithcin equation is needed.

There are another class of SCFTs called Argyres-Douglas theory [54, 55]. It is interesting to see if these theories can also be constructed using six dimensional construction. Our result indicates that it also involves irregular singularity. In fact, irregular singularity can also be used to describe the ordinary superconformal field theory considered in previous chapter.

With this understanding, I hope I can give a fairly complete classification of four dimensional $\mathcal{N}=2$ complete quantum field theories.

A. Irregular solutions to Hitchin's equation

Hitchin's equation is

$$
\begin{aligned}
&F-\phi\wedge\phi=0,\\
&D\phi=D*\phi=0. \qquad (5.1)
\end{aligned}
$$

We want to find local solution to Hitchin's equation and use it to describe $\mathcal{N}=2$ asymptotical free theory as we do for the superconformal case.

Before we start to discussing the irregular solutions, let's discuss the physical meaning of various parameters for Hitchin's equation. In the conformal case, all the gauge coupling constants are dimensionless, they are identified with the complex structure moduli of the Riemann surface. The mass parameters do enter into the description of the Hyperkahler structure, they are the parameters of the coefficient on regular pole, since the the symplectic form in complex structure I depends linearly on these coefficients, which is a requirement for the mass parameter.

In the asymptotical free case, there is a dimensional scale Λ, we can not describe it as the dimensionless complex structure moduli. The dimensional field in the Hitchin's system is the Higgs field (use naive scaling), so this coupling should enter into the definition of the parameter of the Higgs field. The simple pole case is not good since the parameter has been identified with mass parameter. Then we conclude that we need higher order singularity: irregular singularity. The converse is not true since it is still possible to describe superconformal field theory using irregular singularity.

Hitchin's equation does have solutions with irregular singularities. It is the purpose of this chapter to identify what kind of irregular singularities are needed to describe four dimensional asymptotical free $\mathcal{N}=2$ gauge theories. Consider an irregular singularity at the origin, Hitchin's equation is schematically $d\Phi+\Phi^2=0$ for $\Phi=(A,\phi)$, so for solutions singular than $\frac{1}{z}$, they are not compatible unless the solution is abelian, namely, they are taking values in a Cartan subalgebra. The moduli space has similar structure as the regular singular case: it is a hyperkahler manifold; it is an integrable system in complex structure I; the symplectic form depends linearly on the regular pole coefficient, etc. See the detailed explanation in [56], we give a short review in the below.

Introduce local coordinate $z=re^{i\theta}$, and we let t denote the lie algebra of a maximal torus T of the compact lie group G (we take G as SU(N) in this chapter) and t_C its complexification. We pick $\alpha \in t$ and $u_1,...u_n \in t_C$, and consider the solution

$$
\begin{aligned}
A &= \alpha d\theta + ..., \\
\phi &= \frac{dz}{2}(\frac{u_n}{z^n}+\frac{u_{n-1}}{z^{n-1}}+...+\frac{u_1}{z})+\frac{d\bar{z}}{2}(\frac{\bar{u}_n}{\bar{z}_n}+\frac{\bar{u}_{n-1}}{\bar{z}^{n-1}}+...+\frac{\bar{u}_1}{\bar{z}}+...).
\end{aligned} \tag{5.2}
$$

We first assume that u_n is regular and semi-simple, and $u_1=\beta+i\gamma$, when $n=1$, this solution is reduced to the simple pole case. We will later relax regular semi-simple condition for the leading order coefficient.

Let's denote the moduli space of the solution as M_H. The moduli space has the hyperkahler structure and have three distinguished complex structures. In complex strucure I, we get a Higgs bundle for each pair of solutions (ϕ, A). The holomorphic structure of the bundle E is defined by using the $(0,1)$ part of the gauge field A. The $(1,0)$ part Φ of ϕ is a holomorphic section of $ad(E)\bigotimes K_C$. Explicitly, do a complex conjugation using $r^{i\alpha}$, the operator $\bar{\partial}_A = d\bar{z}(\partial_{\bar{z}}+A_{\bar{z}})$ reduces to the standard one $d\bar{z}\partial_{\bar{z}}$. With this trivialization, The holomorphic part of Higgs field is

$$
\Phi = \frac{dz}{2}(\frac{u_n}{z^n}+\frac{u_{n-1}}{z^{n-1}}+...+\frac{u_1}{z}). \tag{5.3}
$$

In another complex structure J, the moduli space does not depend on the complex structure of the Riemann surface, moreover, it is independent of the coefficient in higher order term. We study the G_c valued complex connection $\mathcal{A}=A+i\phi$, which is flat by using of Hitchin's equation. The connection $\mathcal{A}$ can be put in the form

$$
\mathcal{A}_z = (\frac{u_n}{z^n}+\frac{u_{n-1}}{z^{n-1}}+...+\frac{u_2}{z^2}) - i\frac{\alpha-i\gamma}{z}. \tag{5.4}
$$

We put the connection in a standard form:

$$
\mathcal{A}_z = \frac{T_n}{z^n}+\frac{T_{n-1}}{z^{n-1}}+....+\frac{T_1}{z}. \tag{5.5}
$$

T_i depends on the coefficient u_i in an obvious way. For such irregular connection, the monodromy is not just determined by T_1, we have the famous stokes phenomenon and the stokes matrix to describe the so-called generalized monodromy. The dimension of the local moduli space is

$$dim(M_H) = (n)(dim(G_c) - r), \tag{5.6}$$

we will give another way to calculate the dimension in complex structure using Hitchin's fibration. When $n = 1$, this is reduced to the previous formula for the regular pole case with regular-semisimple coefficient. We should emphasize that in defining the moduli space, we fixed the matrices $T_n, ...T_1$. when we apply Hitchin's equation to describe four dimensional gauge theory, T_1 represents as the mass parameters, T_2,...T_n are interpreted as the parameters like dynamical generated scale. The base of the Hitchin's fibration is identified with the Coloumb branch.

What happens when the leading order coefficient is not regular-semisimple? When u_n is semi-simple, the analysis is essentially the same as described in section 6 of [56]. In this case, the subleading order terms $u_{n-1}...u_2$ has some freedom, they are constrained by u_1 but there still some freedom left.

In general, the leading order coefficient can be decomposed into the direct sum as $B_1 \bigoplus B_2 ... \bigoplus B_n$, where some of the blocks are nilpotent, some other blocks are semi-simple, we can then analyze the problem block by block. The semi-simple case is discussed earlier. When u_n is nilpotent, we can also reduce to the semi-simple solution. This is in contrast to the simple pole solution, in that case, when the residue is nilpotent, we have new solutions to the Hitchin equation.

The case with nilpotent leading order coefficient plays an essential role in describing gauge theories, we will give a more detailed review below. Let's consider the case $\mathcal{A}_z = T_n/z^n + ...$, if T_n is nilpotent. We take $G = SL(2, C)$ for an example. $\mathcal{A}_z$

can be written as

$$\mathcal{A}_z = \begin{pmatrix} a & z^{-n}b \\ c & -a \end{pmatrix}, \tag{5.7}$$

where a, c have poles at most of order $n-1$ at $z = 0$ and b is regular. Now by a gauge transformation $g = \begin{pmatrix} 1 & 0 \\ f(z) & 1 \end{pmatrix}$, we can set $a = 0$ by choosing appropriate $f(z)$, the connection becomes

$$\mathcal{A}_z = \begin{pmatrix} 0 & z^{-n}b \\ z^{-k}\tilde{c} & 0 \end{pmatrix}, \tag{5.8}$$

where $\tilde{c}$ is regular and $k < n$. If $n - k > 2$, we can make a further gauge transformation with $g = \begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{\frac{-1}{2}} \end{pmatrix}$ and follow a similar gauge transformation to take the connection back to off-diagonal form. We can reduce n and $n-k$. The only new possibility is then $n = k$ or $n = k+1$, if $n = k$, we are back to the case with T_n regular semi-simple. If $n = k+1$, we take a double cover of a neighborhood around the singular point. We introduce a new coordinate $z = t^2$, then we can reduce to previous situation with a gauge transformation $g = \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & t^{\frac{-1}{2}} \end{pmatrix}$. Write $\mathcal{A} = \mathcal{A}_t dt$, we have $\mathcal{A}_z = \mathcal{A}_t/2t$. We have the following form for the connection

$$A_t = \begin{pmatrix} 0 & t^{-2n}b \\ t^{-2n}c & 0 \end{pmatrix}. \tag{5.9}$$

A_t is even under $t \to -t$, so $A_t dt$ is odd under $t \to -t$. So We have the following singular solution for Hitchin's equation

$$A = 0,$$

$$\phi = \frac{dt}{2}\left(\frac{v_{n-1}}{t^{2(n-1)}} + \frac{v_{n-2}}{t^{2(n-2)}} + ... + \frac{v_1}{t^2}\right) + c.c. \tag{5.10}$$

Now the leading order coefficient is regular semi-simple. It is useful to transform to the original coordinate z, we need to be careful for the transformation, we need to accompany the operation $t \to -t$ with the gauge transformation

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.11}$$

The local solution is

$$\begin{aligned} A = 0, \\ \phi = \frac{dz}{4}\left(\frac{v_{n-1}}{z^{n-1/2}} + \frac{v_{n-2}}{z^{n-3/2}} + ... + \frac{v_1}{z^{\frac{3}{2}}}\right) + c.c. \end{aligned} \tag{5.12}$$

To make this solution well defined, we need to make a gauge transformation M in crossing the cut on the z plane. We can also add the regular terms, the regular terms are of the form $v_k z^{k-1/2}, k \geq 1$ to make the solution well defined, the regular singular term is missing here so this singularity does not encode any mass parameter.

For instance, if $n = 2$, the holomorphic part of the Higgs field (we will call the holomorphic part of the Higgs field as Higgs field in later parts of this paper)is $\Phi_z = \frac{v_1}{z^{3/2}} + \frac{C}{z^{1/2}} + ...$, The spectral curve is $x^2 = \phi_2(z)$, where $\phi_2(z) = Tr(\Phi_z)^2$ is the degree two differential on the Riemann surface. The quadratic differential has the form

$$\phi_2(z) = Tr(\Phi_z)^2 = \frac{q^2}{z^3} + \frac{U}{z^2} + \frac{M}{z} + ..., \tag{5.13}$$

where we take $v_1 = diag(q, -q)$ and $C = diag(a, -a)$. The parameter U depends on a. This parameter is identified with the coulomb branch of the gauge theory, since $\phi_2(z)$ is a degree 2 meromorphic connection, according to Riemann roch theorem, this pole contribute two to the coloumb branch, and so it contributes four to the Hitchin's moduli space. The above method shows how to calculate the local dimension of the moduli space in Hitchin's equation. We expand the spectral curve around the singularity, and find the maximal pole of degree i differential which depends on regular term, and then sum up the contributions from degree 2 differential to degree N differential for $\mathrm{SU}(N)$ case, this gives the local dimension of the base, the total dimension of local Hitchin's moduli space is twice the number we just calculated.

The form of the gauge field and Higgs field can be derived in another straightforward way, we take $n = 2$ for an example. Suppose the Higgs field takes the following form

$$\Phi(z) = \frac{1}{z^2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz + \frac{1}{z}\begin{pmatrix} a & b \\ -c & d \end{pmatrix} dz + ... \tag{5.14}$$

One can calculate the eigenvalues of Φ which are $\frac{1}{z^{3/2}}(\Lambda, -\Lambda)$, and the $\frac{1}{z}$ term are missing because of the monodromy, so we do not have any mass deformation for this type of singularity, this recovers (5.12). The local dimension of the moduli space can be derived by noting that the leading order coefficient belongs to nilpotent orbit with dimension 2 and the regular singularity coefficient is in a semi-simple orbit also with dimension 2, so the local dimension is $2 + 2 = 4$.

More generally, the connection $\mathcal{A}_z$ is an $N \times N$ matrix-valued function with a possible pole at $z = 0$. It has N possibly multiple eigenvalues λ_i. The eigenvalues behave for small z as $z \sim z^{-r_i}$, with rational number r_i. Tame ramification is the case that all r_i are equal to or less than 1. We call completely wild ramification if $r_i > 1$ for all i. The general case is a mixture of these two possibilities. Following SU(2) case, we consider the second order irregular singularity with leading order coefficient nilpotent

$$\Phi(z) = \frac{A_1}{z^2}dz + \frac{A_0}{z}dz + ..., \tag{5.15}$$

where A_1 is the matrix in the nilpotent orbit $\mathcal{O}_1$ labeled by Young Tableaux with partition $[2, 1, 1.., 1]$, we take A_1 as the matrix with standard Jordan form; A_0 is in a regular semi-simple orbit $\mathcal{O}_0$ (the eigenvalues of A_0 are all distinct). One can calculate the eigenvalue of the Higgs field, it has the form

$$\Phi = \frac{1}{z^{1+1/N}} diag(1, \omega, \omega^2 ... \omega^{N-1}) dz, \tag{5.16}$$

where $\omega^N = 1$; Similarly as SU(2) case, $\frac{1}{z}$ term is missing. The local dimension of the Hitchin's moduli space is equal to the sum of the dimension of the orbit $\mathcal{O}_1$ and $\mathcal{O}_0$,

$$d = 2N - 2 + N^2 - N = N^2 + N - 2, \tag{5.17}$$

Notice that this equals to the contribution of a simple regular singularity and a full regular singularity with partition $[n]$. This irregular solution is useful to us since there is only one parameter in the irregular part and this can be identified with the dynamical scale and there is no mass deformation, so this irregular singularity is useful for the pure $\mathcal{N} = 2$ super Yang-Mills theory, we will confirm this in later sections.

We also want to describe theory with any number of fundamental fields, and we are interested in the Higgs field with following eigenvalues (after diagonalization)

$$\begin{aligned} \Phi = & \frac{1}{z^{1+\frac{1}{n-k}}} diag(0, ...0, \Lambda, \Lambda\omega, ...\Lambda\omega^{n-k-1}) dz + \\ & \frac{1}{z} diag(m_1, m_2, ..m_k, m_{k+1}, m_{k+1} ... m_{k+1}) dz + ..., \end{aligned} \tag{5.18}$$

where $\omega = e^{\frac{-2\pi i}{n-k}}$ and the sum of mass vanishes so we have k independent mass parameters.

This form needs a small change for $k = n - 1$, in this case, the Higgs field has the form

$$\Phi = \frac{1}{z^2} diag(\Lambda, \Lambda, .., \Lambda, -(n-1)\Lambda) dz + \frac{1}{z} diag(m_1, m_2, ..m_n) dz + ..., \tag{5.19}$$

$\sum_{i=1}^n m_i = 0$. A special case is when $n = 2$, the leading order coefficient is regular semisimple.

The solution (5.18) is well defined only when we make a gauge transformation on crossing the cut on z plane:

$$M = \begin{pmatrix} 0 & . & . & . & 0 \\ . & 0 & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & 0 & 0 \\ 0 & . & . & . & v_{n-k} \end{pmatrix}, \tag{5.20}$$

where v_{n-k} is the $(n-k)\times(n-k)$ matrix

$$v_{n-k} = \begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & 1 \\ 1 & 0 & 0 & ... & 0 & 0 \end{pmatrix}. \tag{5.21}$$

Let's first consider the simple case $k = 0$, in which the simple pole term is forbidden. We can also add the regular term, however, the regular term must take a form so that the Higgs field is well defined when we cross the cut in the z plane. The Higgs field takes the following form

$$\Phi = \frac{\Lambda}{z^{1+1/n}} diag(1, \omega, ...\omega^{n-1}) + \sum_{d=1}^{n-1} \frac{a_d}{z^{1-d/n}} diag(1, \omega^{-d}..\omega^{-dj}, ...) + ... \tag{5.22}$$

One can check that the Higgs field is well defined using the gauge transformation (5.21) when we cross the cut. Since $(1, \omega, ...\omega^{n-1})$ are the roots of the equation $x^n - 1 = 0$, the equation factorizes as $x^n - 1 = \sum_{i=0}^{n-1}(x - \omega^i)$, expanding the last equation, we have the relation $\sum_i w^i = 0$, $\sum_{i\neq j} \omega^i\omega^j = 0$, $\sum_{i\neq j\neq k} \omega^i\omega^j\omega^k = 0$, etc, the only nonvanishing combination is $\prod w^0 w ... w^{n-1} = (-1)^{n-1}$. We also have the relation $\sum_{j=0}^{n-1} w^{-dj} = 0$ for any d.

Calculating the determinant

$$det(x - \phi(z)) = x^n - \sum_{i=2}^{n} \phi_i(z) x^{n-i}. \tag{5.23}$$

We want to find the leading singular behavior of the coefficient $\phi_i(z)$. For ϕ_2, one may wonder it has the fractional power, but this is not the case, since the coefficient of $\frac{1}{z^{2+\frac{2}{k}}}$ is $\sum_{i \neq j} w^i w^j = 0$, We next calculate the next leading order which depends on the regular term, it has the form

$$\phi_2(z) = \frac{1}{2} \sum_{i \neq j} \omega^i \omega^{-dj} \frac{1}{z^{2+(1-d)/n}} = \sum_i \omega^i \omega^{-di} \frac{1}{z^{2+(1-d)/n}} = \sum_i \omega^{(1-d)i} \frac{1}{z^{2+(1-d)/n}}. \tag{5.24}$$

This term is nonzero only in the case $d = 1$. So $\phi_2(z) = \frac{c}{z^2} + ...$ The calculation can be extended to the other coefficient ϕ_i, there is no term which only depends on the singular term of the higgs field except for $i = n$, the leading order terms depending on the regular terms are

$$\phi_i(z) = C \sum_{n_1 \neq n_2 .. n_i} \omega^{n_1} ... \omega^{n_{i-1}} \omega^{-dn_i} \frac{1}{z^{i+(i-d)/n}} = C \sum_j \omega^{(i-d)j} z^{i+(i-d)/n}. \tag{5.25}$$

We select $n-1$ terms from the irregular term and one regular term, the only vanishing term is when $d = i$, so $\phi_i = C\frac{1}{z^i} + ...$; For the coefficient $\phi_n(z)$, there is a term depending on Λ, it has the form

$$\phi_n(z) = \frac{\Lambda^n}{z^{n+1}} + C\frac{1}{z^n}. \tag{5.26}$$

The contribution of this singularity to the coulomb branch is

$$2 + ...n = \frac{n^2 + n - 2}{2}. \tag{5.27}$$

This is the same as we calculate by counting the dimension of the adjoint orbit for the coefficient on the singular part (5.17).

Let's define the local covering coordinate $z = t^n$, the higgs field has the form

$$\phi(t) = \frac{1}{t^2} diag(1, \omega, ...\omega^{n-1}) dt + ... \tag{5.28}$$

To make this field well defined, we can not turn on the regular singular term. This has the same form as the conventional irregular singularity with leading order coefficient semi-simple.

One can similarly study the spectral curve of the Higgs field (5.18,5.19), the term depending on the regular term is $\phi_i(z) = \frac{C}{z^i}$, so the contribution to the coulomb branch of this singularity is also $\frac{n^2+n-2}{2}$. The difference with the previous case is that we also have the higher order terms with coefficient depending on the singular terms.

To summarize, we have studied several types of irregular singularity: for the first one, the leading order coefficient is regular semi-simple and this kind of singularity is studied extensively; if the leading order coefficient is semisimple but not regular, we can study the moduli space similarly as we do in the case of regular semi-simple leading coefficient; if the leading order singularity is nilpotent, we can also transform it to the first two cases with leading order coefficient semisimple if we go to the covering space of local coordinate patch. if we transform back to the original coordinate, the Higgs field usually has the fractional power in the coordinate.

B. SU(2) theory

In this section, we will identify the corresponding Hitchin system for $\mathcal{N} = 2$ SU(2) gauge theory. One important clue for the Hitchin system is that its total complex dimension must be 2, so the base of the Hitchin fibration is 1 and can be matched with the dimension of the coloumb branch of $\mathcal{N}=2$ SU(2) theory.

There are only two types of irregular singularities: The first one with leading order coefficient semi-simple. The second one has also semi-simple singularity, but it has the form $\frac{1}{z^{n+\frac{1}{2}}}$. We call them type I and type II singularity:

$$\Phi=\frac{1}{z^m}\begin{bmatrix}\Lambda & 0\\ 0 & -\Lambda\end{bmatrix}+\frac{A_{m-1}}{z^{m-1}}+....+\frac{A_1}{z}+..... \tag{5.29}$$

$$\Phi=\frac{1}{z^{m-1/2}}\begin{bmatrix}\Lambda & 0\\ 0 & -\Lambda\end{bmatrix}+\frac{A_{m-1}}{z^{m-1-1/2}}+....+\frac{A_2}{z^{3/2}}+..... \tag{5.30}$$

The contribution of these two singularities to the Hitchin's moduli space is $2m$. The total dimension for the Hitchin's moduli space is the sum of local dimension minus global contribution 6. We use A_n denote type I singularity with $m=n$, and B_n as type II singularity.

There are four asymptotical free theories with only one Coulomb branch: that's SU(2) coupled with $N_f=0,1,2,3$. There are N_f mass parameters and one dynamical scale. To describe these theories, we need Hitchin system with complex dimension 2. The leading order can not exceed over 2, since there is no explanation for the extra parameters in the higher order terms. There are following choices:

$$\begin{aligned}
1:(A_2,B_2) &\quad \text{one mass},\\
2:(A_2,A_2) &\quad \text{two masses},\\
3:(B_2,B_2) &\quad \text{no mass},\\
4:(A_2,A_1,A_1) &\quad \text{three masses},\\
5:(B_2,A_1,A_1) &\quad \text{two masses},\\
6:(A_1,A_1,A_1,A_1) &\quad \text{four masses}.
\end{aligned} \tag{5.31}$$

We recognize the number 6 corresponds to $N_f = 4$ case. In the other cases, there are one dynamical scale (if there are two irregular singularities, one can use scale transformation to make the leading order coefficient equal).

To describe $N_f = 3$, case 4 is the only choice. The Seiberg-Witten curve associated with this Hitchin system is

$$det(x - \Phi(z)) = 0,$$
$$x^2 = \frac{m_1^2}{z^2} + \frac{m_2^2}{(z-1)^2} + \frac{U}{z(z-1)} + \frac{2m_3\Lambda}{z} + \Lambda^2. \tag{5.32}$$

We have used the conformal symmetry to put the simple punctures at $z = 0, 1$ and the irregular puncture at $z = \infty$; U is the Coloumb branch parameter. This is exactly the form (2.36).

Next, let's consider $N_f = 2$ case, case 2 and 5 are possible and we expect them to be equivalent. For case 5, The spectral curve has the form

$$det(x - \Phi(z)) = 0,$$
$$x^2 = \frac{m_1^2}{z^2} + \frac{m_2^2}{(z-1)^2} + \frac{U}{z(z-1)} + \frac{\Lambda^2}{z}. \tag{5.33}$$

We have put two simple singularities at $z = 0, 1$, and the irregular singularity at $z = \infty$. This is the same as listed in (2.36).

For case 2, The spectral curve takes the form

$$det(x - \Phi(z)) = 0,$$
$$x^2 = \frac{\Lambda^2}{z^4} + \frac{m_1\Lambda}{z^3} + \frac{U}{z^2} + \frac{\Lambda m_2}{z} + \Lambda^2. \tag{5.34}$$

We put the puncture at $z = 0, \infty$. We have put two simple singularities at $z = 0, 1$, and the irregular singularity at $z = \infty$. This is the another curve for $N_f = 2$ as listed in (2.36). This proves the equivalence of the above two Hitchin system.

We next need $N_f = 1$, the only choice is case 1, The spectral curve is

$$\begin{aligned} det(x - \Phi(z)) &= 0, \\ x^2 &= \frac{\Lambda^2}{z^3} + \frac{U}{z^2} + \frac{\Lambda m}{z} + \Lambda^2. \end{aligned} \tag{5.35}$$

We also put the puncture at $z = 0, \infty$, which is the same as in (2.36).

We then consider the pure $\mathcal{N} = 2$ SU(2) theory. The only choice is case 3. The spectral curve is

$$\begin{aligned} det(x - \phi(z)) &= 0, \\ x^2 &= \frac{\Lambda^2}{z^3} + \frac{U}{z^2} + \frac{\Lambda^2}{z}. \end{aligned} \tag{5.36}$$

The Seiberg-Witten curve above is the same as in (2.36).

The above analysis exhausted $\mathcal{N} = 2$ SU(2) gauge theories with fundamental hypermultiplet. We should remark that there are more choices for the Hitchin moduli space with total dimension 2. For a singularity with regular semi-simple coefficient, the local dimension is $2n$, where n is the order of the singularity, this includes the regular singularity. If the leading coefficient is nilpotent, as we showed in last section, the dimension is also $2n$, if the leading order singularity has the form $\Phi = \frac{v_n}{z^{n-1/2}}dz, n = 2...$ We have the following choices except those we studied in this section:

i) One 3 order irregular singularity with regular semi-simple coefficient and one regular singularity, we have two mass parameters associated with the residue of the regular singularity. This might be related to A_2 Argyres-Douglas fixed point [54, 7].

ii) One 3 order irregular singularity with the form (5.12) with $n = 3$, and a simple singularity, we suspect this is related to A_1 Argyres-Douglas fixe point

iii) One 4 order irregular singularity with regular semi-simple coefficient, we suspect this is also related to A_1 Argyres-Douglas fixed point.

iv) One 4 order irregular singularity with the form (5.12), we suspect this is associated with the A_0 Argyres-Douglas superconformal fixed point.

There are some clues that the above conjecture might be true. Singular fibre is classified by Kodaira, and Argyres-Douglas fixed point corresponds to singular fibre of type A_2, A_1 and A_0. According to the result by Boalch [57], case i) can be associated with the affine dynkin diagram of A_2, case ii) is associated with affine dynkin diagram of A_1 and case iv) is associated with dynkin diagram of A_0. There are another hint about our conjecture, for A_2 Argyres-Douglas fixed point, we have two deformation paramters, we also have two deformation parameters in the Hitchin system i), one from the order 3 singularity and the other from regular singularity. Case ii) and case iii) both have one mass parameter and case iv) does not have mass parameter which match the deformation parameter of A_1 and A_0 singularity. This conjecture is true and is proved in Chapter VI.

It is natural to then consider the linear quiver with only SU(2) gauge groups. For the superconformal case with n SU(2) gauge group, we have a total of $n+3$ punctures on the sphere, with $n-1$ punctures to account for the flavor symmetry of the bi-fundamental and 2 puncture for the two fundamentals on the far left, and 2 punctures to account for the bi-fundamental on the far right. If the quiver is not conformal, we can only change the number of fundamentals at the end, we need to replace the two simple punctures with the irregular puncture based on solution (5.18,5.19) with $n = 2, k$, where k is the number of fundamentals on the end, see Figure 17 for details. If $k = 2$, we still have two simple punctures. We have a total of $n+1$ punctures on the sphere if $k < 2$ on both ends, and we have $n-2$ gauge groups which are conformal; the punctured sphere has a total of $n-2$ moduli and these moduli are identified with the UV gauge couplings of the $(n-2)$ conformal gauge group.

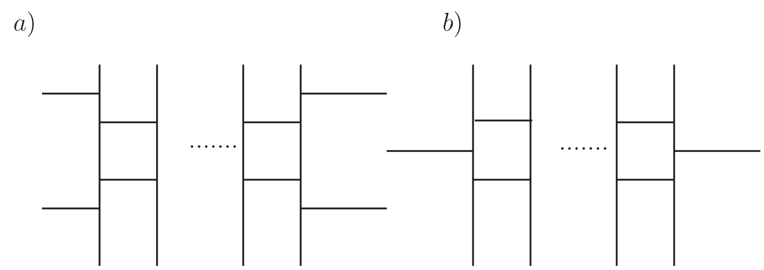


Fig. 17. Left: Brane configuration of conformal SU(2) quivers. Right: A brane configuration with non-conformal gauge group.

In conclusion, to describe SU(2) linear quiver, besides the regular singularity, we also need to turn on two types of irregular singularities. One can similarly studied the different degeneration limits and we will get generalized quiver as in [11]. For example, let's consider the quiver in Figure 18a). There are three SU(2) gauge groups and only the middle one is conformal. The weakly coupled limit of quiver Figure 18a is described by the degeneration limit of the Riemann sphere with four punctures in Figure 18b: we have two simple punctures and two irregular punctures described by boxes. Figure 18c) describes another degeneration limit of the same Riemann sphere, after the complete degeneration limit, we get a theory which is described by Hitchin's equation with two irregular singularities and one regular singularity, this is depicted in Figure 18d). This theory has two dimension two operators in Coulomb branch and it is a linear quiver with two SU(2) gauge group. The dual theory is a generalized quiver.

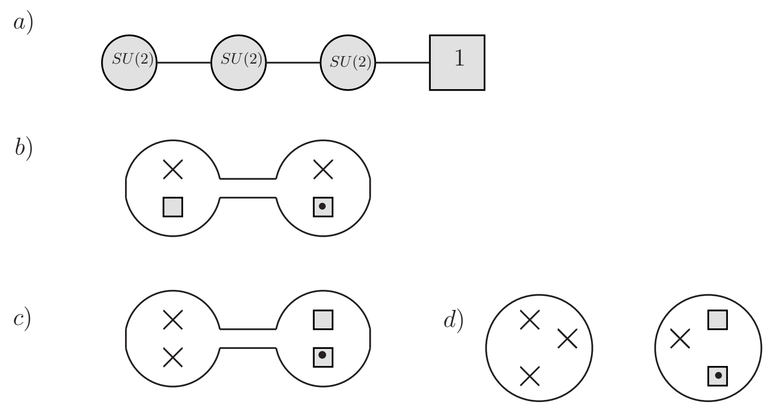


Fig. 18. a): A nonconformal SU(2) quiver with three gauge groups. b): The degeneration limit of the Riemann sphere corresponding to quiver a), the regular singularity is represented by cross, and the irregular singularity is represented by box. c): Another degeneration limit of the same Riemann sphere. d): After complete degeneration, we get a SU(2) theories with two irregular singularity and a simple singularity.

C. SU(N) theory

1. One SU(N) gauge group

Let's now generalize the analysis of SU(2) theory to SU(N) case. We first consider a single SU(N) gauge group with N_f fundamental hypermultiplets. We already know how to describe $N_f = 2N$ case in six dimensional language. We have four punctures on the sphere: two simple regular punctures and two full regular punctures. We can think that one simple puncture and one full puncture are needed to describe N fundamental on each side. The contribution of these two punctures to Coulomb branch parameters is $N-1+\frac{1}{2}(N^2-N) = \frac{N^2+N-2}{2}$, where $N-1$ is the contribution of the simple regular puncture and $\frac{1}{2}(N^2-N)$ is the contribution of the full regular puncture.

To describe asymptotical free theories. The total dimension of Hitchin's moduli space should be $2(N-1)$. We need just one dynamical generated scale and N_f mass parameters. The Hitchin system is not unique as we can see from SU(2) example and the Seiberg-Witten curve in the form (2.35) (the decomposition of the mass terms are not unique and are isomorphic). There can not be more than two irregular singularities, since there is only one dynamical generated scale. The maximal number of singularity is three, since there is a marginal coupling for the four singularity case which is impossible in our case.

The leading order in the irregular singularity should has only one nilpotent block, and the nilpotent block should have the form $(n-k,1,1,1,1...1)$, after diagonalization, the detailed form depends on the subleading term. It turns out that the leading order eigenvalue has the form $\frac{1}{z^{1+\frac{l}{k}}}(1,\omega,...\omega^{n-k-1},0,0,0..0)$, where $1\leq l<k$. There are k mass parameters in this singularity. Let's first consider two irregular singularities case. The local contribution to Coulomb's branch can be calculated, the minimal number is $\frac{N^2+N-2}{2}$ for $l=1$. So the only possibility is $l=1$ for two singularities, and this can be used to describe SU(N) with $N_f=k_-+k_+$. Now let's consider three singularities, as from the local dimension of irregular singularity, the extra two singularity must be the regular singularity. Moreover, one can prove that the two regular singularity must be a regular full and a regular minimal, since only in this case, there is a new regular full puncture which can be used to couple the SU(N) gauge theory. Since there are a total of N mass parameters in two regular singularities, the three singularity cases is only possible for $N_f\geq N$.

The above result can be explained using brane construction; In analogy with SU(2) theories, we first consider $2N>N_f\geq[\frac{1}{2}N]$, in this case we can put N fundamentals on the right and N_f-N fundamentals on the left, and we must have a simple and a full regular punctures to describe the fundamentals on the right. We can only have one irregular punctures to account for the fundamentals on the left as SU(2) case. To get the correct number of coulomb branch moduli, the contribution of the irregular singularity to Coulomb moduli space must be $\frac{1}{2}(N^2-N)$; We do have a class of irregular singularity with this number in (5.18,5.19). From the analysis of SU(2) theory, we may want to select the solution with $n=N, k=N_f-N$, an important check is that the flavor symmetry on the $N-N_f$ fundamentals on the left hand side is U(k). The regular singular part of the irregular singularity has the partition $(k+1,1,1,...1)$, which do describe U(N) flavor symmetry.

We now have the clue to describe SU(N) theory with any number of fundamentals. We can decompose $N_f=k_-+k_+$ and $k_-<k_+\leq N$, namely, we put k_- semi-infinite D4 branes to the left and k_+ semi-infinite D4 branes to the right. See the brane configuration in Figure 19. If $k_+<N$, we need two irregular singularities, and the local solution is of the form (5.18,5.19) with $n=N, k=k_-$ and $n=N, k=k_+$; if $k_+=N$, we have two regular punctures and one irregular puncture with $n=N, k=k_-$. We also need to set the coefficient Λ at the irregular singularities equal. The Seiberg-Witten curve is derived from the spectral curve of the Hitchin system. Notice that, we have more than one description for the same SU(N) theory with N_f fundamentals. The different Hitchin moduli spaces corresponding to different decomposition of N_f are isomorphic! see [13, 58].

Let's consider pure SU(N) theory for an example. We need to have two irregular singularities with $n=N, k=0$ in (5.18,5.19). The spectral curve is described in (5.25). We put two singularities at $z=0, z=\infty$, and the Seiberg-Witten curve is

$$x^N+\frac{u_2}{z^2}x^{N-2}+\frac{u_3}{z^3}x^{N-3}+....+\frac{u_{N-1}}{z^{N-1}}x+\frac{\Lambda^N}{z^{N+1}}+\frac{u_N}{z^N}+\frac{\Lambda^N}{z^{N-1}}=0. \tag{5.37}$$

The Seiberg-Witten differential is $\lambda=xdz$.

The solution of pure SU(N) theory is related to another integrable system: peri-

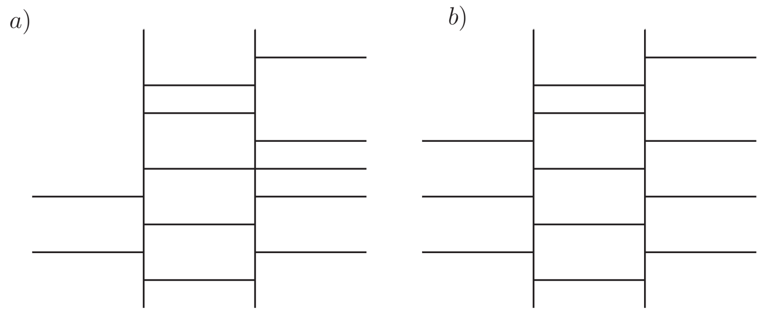


Fig. 19. Left: A brane configuration for SU(5) theory with 7 fundamentals, here $k_- = 2, k_+ = 5$. Right: Another brane configuration for the same theory as a), here $k_- = 3, k_+ = 4$.

odic Toda chain [59]. Here we give another integrable system to describe pure SU(N) theory using Hitchin's system, these two integrable systems should be isomorphic.

The specific form of the singularity of the solution to Hitchin's equation may be seen from the Brane configuration and Seiberg-Witten curve. The Seiberg-Witten curve for the brane configuration in Figure 19 is

$$F(v,t) = c_0 \prod_{i=1}^{k_-}(v-m_i)t^2 + B(v)t + c_2 \prod_{i=1}^{k_+}(v-m_i) = 0, \tag{5.38}$$

where $B(v) = c_1(v^N + u_2 v^{N-2} + \ldots u_N)$, and the Seiberg-Witten differential is $\lambda = \frac{v}{t}dt$. We regard this curve as the polynomial in v with fixed t, so we have a total of N roots. In the limit $t \to 0$, k_+ roots are constant, they are $m_1, m_2, \ldots m_{k_+}$, and we have $N-k_+$ roots which are $\Lambda t^{\frac{1}{N-k_+}}(1, \omega, \ldots \omega^{N-k_+-1})$, where ω is the root of the equation $x^{N-k_+} = 1$, and Λ depends on c_α.

Now we want to find the Hitchin system description. The Seiberg-Witten curve is identified with the spectral curve of Hitchin system

$$det(x - \Phi(z)) = 0, \tag{5.39}$$

and the Seiberg-Witten differential is $\lambda = xdz$. Let's compare this with (5.38), we identify z with t, and $x = \frac{v}{t}$, we have the equation

$$det(\frac{v}{t} - \Phi(t)) = t^N det(v - t\phi(t)) = F(v,t). \tag{5.40}$$

Now we can read the boundary condition of the Higgs field at the puncture $t = 0$, The roots of v at fixed t is identified with the eigenvalue of the function $t\Phi(t)$, so the Higgs field has the following form at $t = 0$,

$$\begin{aligned}\Phi = &\frac{1}{t^{1+\frac{1}{N-k_+}}} diag(0,...0,\Lambda,\Lambda\omega,...\Lambda\omega^{N-k_+-1})dt + \\ &\frac{1}{t} diag(m_1, m_2, ..m_k, m_{k+1}, m_{k+1}...m_{k+1})dt + ...\end{aligned} \tag{5.41}$$

We do a little bit manipulation on the simple pole term so that the matrix is traceless, i.e. they are taking value in the lie algebra of SU(N). This is the exactly same as (5.18, 5.19). In the case $k_+ = N$, the irregular term is absent and we have N mass terms, however, the maximal parameter for a regular singularity is $N - 1$, so we need to turn on another simple singularity with only one mass parameter. The analysis can be carried similarly for $t = \infty$.

2. Asymptotically free quiver gauge theory

a. Review for the conformal case

We next turn to the description of the quiver gauge theory. The simplest case is a quiver with all SU(N) gauge groups, and only the two SU(N) group at the ends are not conformal. If we have n SU(N) gauge groups, then there are $n - 1$ simple punctures and two irregular punctures depending on the number of fundamentals on the end SU(N) gauge group. The dimension of the complex structure moduli space of this sphere with $n+1$ punctures is $n-2$, this matches the number of conformal gauge group and is used to describe the UV conformal gauge coupling constants. If there are k_- fundamental hypermultiplets on the far left SU(N) gauge group and k_+ fundamentals on the far right, the two irregular singularity is of the form (5.18, 5.19) with $n = N, k = k_\pm$.

Let's consider the general quiver gauge theory with the gauge group $\prod_{i=1}^{n} SU(k_i)$, here $k_1 < k_2 ... < k_r = ...k_s > k_{s+1} .. > k_n$ and $k_s = ..k_r = N$, the rank of the gauge group is chosen so that every gauge group is conformal or asymptotically free; we have the bi-fundamental fields in adjacent gauge groups, we also add d_i fundamental hypermultiplets to ith gauge group $SU(k_i)$. The $r-s-1$ middle SU(N) gauge groups are conformal. The case with all gauge group conformal is studied in [11, 23] we want to extend to general asymptotical free case. We review the superconformal theory which will provide us a lot of clues, in this case $d_\alpha = 2k_\alpha - k_{\alpha-1} - k_{\alpha+1}$.

The Seiberg-Witten curve for this theory is derived by lifting the brane configuration to M theory. The $D6$ branes are described by Taub-NUT space [9]. $NS5-D4$ brane configurations become a single M5 brane embedded in D6 branes background. Define coordinate $v = x^4 + ix^5$ and polynomials:

$$J_s = \prod_{a=i_{s-1}+1}^{i_s} (v - m_a), \tag{5.42}$$

where $1 \leq s \leq n$ and $d_\alpha = i_\alpha - i_{\alpha-1}$, m_a is the constant which represents the position of D6 brane in v direction and is identified with the mass of the fundamental hypermultiplet.

The Seiberg-Witten curve is

$$t^{n+1}+g_1(v)t^n+g_2(v)J_1(v)t^{n-1}+g_3(v)J_1(v)^2J_2(v)t^{n-2}$$
$$+...+g_\alpha\prod_{s=1}^{\alpha-1}J_s^{\alpha-s}t^{n+1-\alpha}+...+f\prod_{s=1}^{n}J_s^{n+1-s}=0, \tag{5.43}$$

here g_α is a degree k_α polynomial of variable v. From the study of a single SU(N) theory, to get a Hitchin description, it is necessary to move all the D6 branes to the far left and far right. We split J_α as the product of $J_{\alpha,L}$ and $J_{\alpha,R}$, where $J_{\alpha,L}$ denotes the D6 branes moving to the left, and $J_{\alpha,R}$ denotes the D6 branes moving to the right. We can choose $J_{\alpha,L}$ and $J_{\alpha,R}$ arbitrarily. We define the canonical choice by moving all the fundamental matters for $SU(k_i), i\leq r$ to the far left, and move all the fundamental matters for $SU(k_i), i\geq s$ to the far right. In the case of $r=s$, we split the fundamentals into two parts $d_{r,L}=N-k_{r-1}$ and $d_{r,R}=N-k_{r+1}$.

After moving the D6 branes to the infinity, the Seiberg-Witten curve becomes

$$F(v,t)=\sum_{\alpha=0}^{n+1}\hat{g}_\alpha(v)t^{n+1-\alpha} \tag{5.44}$$

where

$$\hat{g}_\alpha(v)=c_\alpha g_\alpha(v)\prod_{\beta=\alpha+1}^{r}J_\beta^{\beta-\alpha} \qquad \alpha=0,1,....r-1. \tag{5.45}$$

$$\hat{g}(v)=c_\alpha g_\alpha(v), \qquad \alpha=r,....s. \tag{5.46}$$

$$\hat{g}(v)=c_\alpha g_\alpha(v)\prod_{\beta=s}^{\alpha}J_\beta^{\alpha-\beta} \qquad \alpha=s-1,.....n. \tag{5.47}$$

Let's calculate the order of the polynomial $\hat{g}_\alpha(v)$. The middle one is not changed. There are two quiver legs and we study one of them and the other leg can be treated similarly. Let's consider the leg $SU(k_1)-SU(k_2)-...-SU(k_r)$ with $k_r=N$, We can associate a Young Tableaux to this leg with the rows $n_1=k_1$, $n_2=k_2-k_1$,...,$n_r=k_r-k_{r-1}$, it is easy to see that the total box of the Young Tableaux is N. Then the number of fundamentals can be written as $d_\alpha = 2k_\alpha - k_{\alpha-1} - k_{\alpha+1} = (k_\alpha - k_{\alpha-1}) - (k_{\alpha+1} - k_\alpha) = n_\alpha - n_{\alpha+1}$. $J_\alpha(v)$ is a order d_α polynomial in v. The order of polynomial $\hat{g}_\alpha(v)$ is

$$d(g_\alpha) = k_\alpha + d_{\alpha+1} + 2d_{\alpha+2} + ...(r-\alpha)d_r = N. \tag{5.48}$$

The same calculation can be applied to the right quiver leg, so the polynomial $g_\alpha(v)$ has the same order N. We want to study the roots of the Seiberg-Witten curve regarded as a polynomial in v while t is fixed. v have n constant roots at the roots of the following equation

$$\sum_{\alpha=0}^{n+1} c_\alpha t^{n+1-\alpha} = 0. \tag{5.49}$$

This polynomial is the the coefficient of v^N in $F(v,t)$ if we regard it as the polynomial in v. The singular behavior of the Higgs field of the Hitchin equation can be derived from the roots as we did for a single SU(N) gauge group theory, the Higgs field has regular simple singularity at $t = t_\alpha$ which is the root of above equation. The Higgs field has the form

$$\Phi(z) = \frac{1}{z} diag(\underbrace{m, ...m}_{N-1}, -(N-1)m)dz + ... \tag{5.50}$$

There are other singularities for the Hitchin system, we study the roots of the Seiberg-Witten curve at $t \to \infty$, the n roots of v are dictated by the polynomial $g_0(v)$,

$$v = (\underbrace{m_{d_1,1}, m_{d_1,2}, ..m_{d_1,d_1}}_{d_1} \underbrace{m_{d_2,1}, m_{d_2,1}, ...m_{d_2,d_2}, m_{d_2,d_2}}_{2d_2}, ...). \tag{5.51}$$

The Higgs field is therefore

$$\Phi(z) = \frac{1}{z} diag(\underbrace{m_{d_1,1}, m_{d_1,2}, ..m_{d_1,d_1}}_{d_1} \underbrace{m_{d_2,1}, m_{d_2,1}, ...m_{d_2,d_2}, m_{d_2,d_2}}_{2d_2}, ...)dz + ... \tag{5.52}$$

It is interesting to note that the mass pattern can be derived from the Young Tableaux.

Indeed, it can be read from the dual Young Tableaux. To construct dual Young Tableaux, we simply exchange the rows and columns. More precisely, the dual Young Tableaux is related to the original Young Tableaux as follows: we have a total of n_1 rows, we have $n_1 - n_2$ rows with length 1, $n_k - n_{k+1}$ rows with length k, etc; we use the convention $n_{r+1} = 0$. Look at the form of the Higgs field, we see that the residue is dictated by dual Young Tableaux, we have $d_1 = n_1 - n_2$ mass parameters with degeneracy 1, we have $d_2 = n_2 - n_3$ mass parameters with degeneracy 2, etc. The same analysis applies to the case $t \to 0$, so the theory is described by a Riemann sphere with $n+1$ simple punctures and two generic punctures.

b. Irregular puncture for nonconformal theory

When not all $d_i = (2k_i - k_{i-1} - k_{i+1})$ are satisfied, the quiver gauge theory is non-conformal. Let's first consider the case where there is no fundamentals for the left tail. The regular singularities describing the middle part and the right tail should not be changed. There should only have just an order two irregular singularity needed to describe left-tail, since otherwise there have an extra marginal coupling which is inconsistent with gauge theory. There are a total of $r-1$ mass parameter for the bi-fundamental, there are r dynamical generated scale. The leading order singularity should have r nilpotent block and each nilpotent block should be the maximal form, i.e. whose partition is $[n_i]$. The constraint on this irregular singularity is that its local dimension should give the correct dimension for the Coulomb branch of this tail.

Let's first calculate the dimension for the minimal case, this means that the eigenvalues for each block are $\frac{\Lambda_i}{z^{1+\frac{1}{n_i}}}(1, \omega, ..\omega^{n_i-1})$ with $\omega^{n_i} = 1$. The spectral curve around the irregular singularity is

$$\det(x - \phi(z)) = \prod_i^r (x^{n_i} + \frac{f(m_i)}{z} x^{n_i-1} + \frac{u_2}{z^2} x^{n_i-2} + ... \frac{(\Lambda_i)^{n_i}}{z^{n_i+1}} + \frac{u_n}{z^{n_i}}). \tag{5.53}$$

Expanding the spectral curve as $x^N = \phi_i(z)x^{N-i}$, and find the maximal order of pole of ϕ_i which do not solely depend on Λ and m, we start with $N - n_1 < j \leq N$, we choose the term from n_1 factor and constant terms from other factors, the orders of pole of ϕ_j are given by

$$ord(\phi_{N-n_1+1}) = \sum_{i=2}^{r} n_i + r - 1, \quad ord(\phi_{N-n_1+2}) = \sum_{i=2}^{r} n_i + r - 1 + 2, ...$$
$$ord(\phi_{N-1}) = \sum_{i=2}^{r} n_i + r - 1 + n_r - 1, \; ord(\phi_N) = \sum_{i=2}^{r} n_i + r - 1 + n_r. \tag{5.54}$$

The same analysis can also be carried out for $N - \sum_{i=1}^{k} n_i < j \leq N - \sum_{i=1}^{k-1} n_i$ with $1 \leq k \leq r$, and ϕ_j has the following orders of pole

$$\sum_{i=k+1}^{r} n_i + r - k, \sum_{i=k+1}^{r} n_i + r - k + 2, ..., \sum_{i=k+1}^{r} n_i + r - k + n_k. \tag{5.55}$$

So the total dimension is

$$\sum_{k=1}^{r} [\sum_{i=k+1}^{r} n_i + r - k + \sum_{i=k+1}^{r} n_i + r - k + 2 + ... + \sum_{i=k+1}^{r} n_i + r - k + n_k]. \tag{5.56}$$

After some calculation, the above expression becomes

$$\frac{1}{2}\sum_{k=1}^{r} n_k^2 + \sum_{k<i} n_i n_k - r + \sum_{k=1}^{r}(r - k + \frac{1}{2})n_k. \tag{5.57}$$

It is interesting to write the Coulomb branch dimension of the quiver tail in terms of the Young Tableaux. In the conformal case, it is encoded by one generic regular singularity and r simple regular singularities, the total Coloumb branch dimension from these punctures is

$$\frac{1}{2}[N^2 - \sum_i r_i^2 + r(2N - 2)], \tag{5.58}$$

where r_i is the height of the ith column of Young Tableaux. We would like to express it in terms of the rows of the Young Tableaux; This can be done by noting that we have $n_k - n_{k+1}$ columns with height k, and the above formula becomes

$$\frac{1}{2}\sum_i n_i^2 + \sum_{i<j} n_i n_j - r + \sum_i (r - i + \frac{1}{2}) n_i. \tag{5.59}$$

Comparing formula (5.57) and (5.59), we immediately conclude that the irregular singularity has the same partition as the Young Tableaux of the left quiver tail. We also need to match the number of dimension i Coulomb branch parameter. An important difference for irregular singularity is that not all the dimension i Coulomb branch parameters are encoded in the coefficient ϕ_i. In fact, there are $r-1$ extra dimension 2 operators encoded in ϕ_N.

When the fundamental matters are added, the best way to find the irregular singularity is through the Seiberg-Witten curve which we leave it to later part of this section.

We need to clarify some of the special cases, we have at least $r-s-1$ simple singularities which are used to describe the bi-fundamentals between the SU(N) group. We may have more simple singularities if some of the gauge groups on the quiver tail is conformal. If $n_{\alpha-1} - \sum_{i=\alpha-1}^{r} d_i \neq 0, n_\alpha - \sum_{i=\alpha}^{r} d_i = 0$ for some α, one can show that all gauge groups SU(k_i) with $r \geq i \geq \alpha$ are conformal, and only $\alpha - 1$ blocks in the Higgs field are irregular, we need to add $r - \alpha + 1$ more simple singularities to account for the mass deformation of the bi-fundamental fields; this can also be seen from the fact that we have more roots for the equation before v^N term in $F(v,t)$. If $\alpha = 1$ for the above situation, we return to the superconformal case. There is another justification to add more simple singularities, since we only have α dynamical scale from the irregular singularity from the irregular singularity, but we have extra $r - \alpha + 1$ UV dimensionless gauge couplings, these can only be represented by the complex structure moduli of the Riemann surface, so we need to add $r - \alpha$ simple regular punctures. We also lose mass parameters m_α for each regular block, these parameters are now encoded in the simple regular punctures.

What happens if there is a gauge group $\mathrm{SU}(k_\beta)$ which is conformal, but $\beta < \alpha$, where for α,

$$n_{\alpha-1} - \sum_{i=\alpha-1}^{r} d_i \neq 0, \quad n_\alpha - \sum_{i=\alpha}^{r} d_i = 0. \tag{5.60}$$

The above analysis implies that we need an irregular singularity and $(r - \alpha + 1)$ simple regular singularities to describe the quiver tail. We want to identify UV gauge couplings for $\mathrm{SU}(k_\beta)$, it is not represented by the complex structure moduli of the punctured Riemann sphere, it is encoded in the irregular part of the irregular singularity. The condition for the conformal gauge coupling of $\mathrm{SU}(k_\beta)$ is $d_\beta = n_\beta - n_{\beta+1}$, the number of non-zero entries in irregular part of v_{n_β} and $v_{n_{\beta=1}}$ are

$$\begin{aligned} r_\beta &= n_\beta - \sum_{i=\beta}^{r} d_\beta, \\ r_{\beta+1} &= n_{\beta+1} - \sum_{i=\beta+1}^{r} d_\beta. \end{aligned} \tag{5.61}$$

The conformal condition for $\mathrm{SU}(k_\beta)$ implies $r_\beta = r_{\beta+1}$. Now the dimensionless gauge coupling for $\mathrm{SU}(k_\beta)$ is identified with $\tau_\beta = \frac{\Lambda_\beta}{\Lambda_{\beta+1}} = \frac{c_{\beta-1}c_{\beta+1}}{c_\beta^2}$. This case shows that we can encode the conformal couplings in the irregular singularity, while in the canonical treatment of superconformal field theory, the gauge coupling is encoded as the complex structure moduli of the Riemann surface. In fact, one can also encode all the dimensional gauge couplings of the superconformal field theory into the irregular singularity.

The form of the irregular puncture can also be derived from the Seiberg-Witten curve, here the number of fundamentals are arbitrary and constrained by the relation $d_i \leq (2k_i - k_{i-1} - k_{i+1})$. The Seiberg-Witten curve is the same as (5.44), the difference is that here c_α is dimensional parameters to make every term in $F(v,t)$ have the same dimension. Similarly, we have the simple singularity for the Higgs field at the points t_α which are the roots of the polynomial of the coefficient of the v^N term. There are other two singularities at $t=0$ and $t=\infty$, these two describe the right quiver tail and left quiver tail respectively. We focus on the left tail. In the case $r=s$, there is no canonical way to split the fundamental matters on the $SU(k_r)$ node, but we have to make sure that $d_{rL} \leq N-k_{r-1}$, $d_{rR} \leq N-k_{r+1}$.

The order of coefficient for $\hat{g}_\alpha(v)$, $\alpha<r$ is

$$deg(\hat{g}_\alpha(v)) = k_\alpha + d_{\alpha+1} + 2d_{\alpha+2}....(r-\alpha)d_r = \hat{k}_\alpha, \tag{5.62}$$

k_α is a non-decreasing series and we have

$$\hat{k}_\alpha - \hat{k}_{\alpha-1} = (k_\alpha - k_{\alpha-1}) - \sum_{i=\alpha}^{r} d_i = n_\alpha - \sum_{i=\alpha}^{r} d_i. \tag{5.63}$$

We take $\hat{k}_{\alpha-1}=0$ and we have the condition $\hat{k}_\alpha - \hat{k}_{\alpha-1} \geq 0$, notice that $\hat{k}_r = N$.

In the limit of $t\to\infty$, v have $\hat{k}_0$ constant roots

$$v = (\underbrace{m_{d_1,1}, m_{d_1,2}, ..m_{d_1,d_1}}_{d_1} \underbrace{m_{d_2,1}, m_{d_2,1}, ...m_{d_2,d_2}, m_{d_2,d_2}}_{2d_2}, ...). \tag{5.64}$$

Since $\hat{k}_0 < N$, there are other roots besides the constant ones we get above, for any $\alpha \leq r$, we have the $\hat{k}_\alpha - \hat{k}_{\alpha-1}$ roots

$$v = \Lambda_\alpha t^{\frac{1}{m}}(1, \omega, ...\omega^{m-1}), \tag{5.65}$$

where $m = \hat{k}_\alpha - \hat{k}_{\alpha-1}$, ω is the root for $x^m=1$ and $\Lambda_\alpha = (\frac{c_{\alpha-1}}{c_\alpha})^{\frac{1}{m}}$, one can check that Λ has dimension 1 from the Seiberg-Witten curve $F(v,t)$, if we require t has dimension -1 and v has dimension 1.

Based on the roots of v in the limit $t\to\infty$, the Higgs field has the following form around the singularity $z^{'}=\infty$ (we change the local coordinate to $z=\frac{1}{z'}$

$$\phi(z)=\begin{pmatrix} v_{n_1} & 0 & 0 & 0 & 0 \\ 0 & v_{n_2} & 0 & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & v_{n_r} \end{pmatrix} dz+...,\qquad (5.66)$$

where v_{n_α} is the diagonal matrix as in (5.18) with $n=n_\alpha, k=\sum_{i=\alpha}^{r} d_i$, to make the Higgs field well defined, we make a gauge transformation of the block diagonal form (5.21) when we cross the cut. The mass terms in v_{n_α} is constrained though, its form is

$$\frac{1}{z}diag(m_{d_\alpha,1},m_{d_\alpha,2},...,m_{d_\alpha,d_\alpha},....,m_{d_r,1},m_{d_r,2},...,m_{d_r,d_r}).\qquad (5.67)$$

Namely, the mass parameter in d_1 has degeneracy 1, the mass parameter in d_2 has degeneracy 2, etc.

It is interesting to compare the total mass parameters with the quiver tail. The mass parameters $(m_1,m_2,..m_{d_1}...)$ are used to describe the mass deformation for the fundamentals, and we have a total of $r-1$ mass parameter m_α (One of m_α is eliminated by traceless condition), and we have a total of $(r-1)$ bi-fundamental matter fields, so the mass parameters match the matter contents of the quiver gauge theory.

If $n-k=1$ for v_{n_α}, we need a little bit modification, the Higgs field is (we assume $n_r-k=1$ here for an illustration).

$$\Phi(z)=\begin{pmatrix} v_{n_1}+\frac{1}{z^2}\Lambda_r I_{n_1\times n_1} & 0 & 0 & 0 \\ 0 & v_{n_2}+\frac{1}{z^2}\Lambda_r I_{n_2\times n_2} & 0 & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & 0 & \frac{-(n-1)}{z^2}\Lambda_r \end{pmatrix} dz+...\qquad (5.68)$$

The coefficient Λ_i is identified with the dynamical scale of ith gauge group $SU(k_i)$.

In summary, for A type $\mathcal{N}=2$ quiver gauge theory, the six dimensional description involves several regular singularities and two irregular singularities. In the superconformal case, the irregular singularity becomes also the regular singularity and they are of the special type which is dictated completely by the rank of the gauge group. In the non-conformal cases, the irregular singularities are determined by the rank of the gauge group and the number of the fundamentals. They can be described uniformly.

Similarly, for the non-conformal quiver, one can study different degeneration limits and study different dual frame of the same theory. In the completely degeneration limit, we can find new theories without conventional lagrangian description. There are a lot more irregular singularities which deserve further study.

CHAPTER VI

THREE DIMENSIONAL MIRROR SYMMETRY *

The moduli space of Hitchin's equation is a hyperkahler manifold. In one complex structure, there is a Hitchin's fibration, and the spectral curve is identified with the Seiberg-Witten curve. The whole hyperkahler manifold is naturally identified with the Coulomb branch of the corresponding four dimensional theory on $R^3 \times S$.

For three dimensional $\mathcal{N} = 4$ theory, it usually has Coulomb branch and Higgs branch. There is a remarkable mirror symmetry of three dimensional theory [60]: for a theory A, there is another theory B whose Higgs branch is identified with the Coulomb branch of A and vice versa. The theory A I want to study is the compactified theory of the four dimensional theory studied in last two sections, it is interesting to find their three dimensional mirror theory B. This would provides new examples of three dimensional mirror symmetry. If the mirror theory has lagrangian description, the property of four dimensional theory A which usually does not have a lagrangian description will be better understood.

Let's look more closely at our torus example with x^3 compact and get a three dimensional theory. It makes sense to talk about electric description and magnetic description when we talk about three dimensional theory. The electric description has dynamical $M5$ branes (from D4 branes) wrapped on coordinate x^3, x^6, x^{10} and other $M5$ branes wrapped on x^3, x^4, x^5. In the magnetic description, there are dynamical $M5$ branes wrapped on x^3, x^6, x^10 and $M5$ branes wrapped on x^4, x^5, x^10. So we see

*Part of the result reported in this chapter is reprinted with permission from *Mirrors of 3d Sicilian theories* by F. Benini, Y. Tachikawa and D. Xie, published in JHEP **1009** (2010) 063, Copyright [2010] by SISSA; *More three dimensional mirror pairs* by D. Nanopoulos and D. Xie, published in JHEP **1105** (2011) 071, Copyright [2011] by SISSA.

that the magnetic description is derived by exchanging the coordinate x^3 and x^{10}. This is just the $3d$ mirror symmetry. The mirror is not a 3d gauge theory though, in fact, we need to take the radius of x^3 infinitely small and flow to the IR, we expect the mirror is a three dimensional theory.

Let's focus on the dynamical $M5$ branes, the electric description can be derived by first compactified on a torus (x^3, x^{10}), and we get a four dimensional $\mathcal{N}=4$ theory, and then compactify on x^6 to get a three dimensional theory, the four dimensional gauge coupling is the complex structure of the torus (x^3, x^{10}). The magnetic description is just exchange x^3 and x^{10}, from the gauge theory point of view, this is just the electric-magnetic duality. Then 3d mirror symmetry is coming from four dimensional S-duality. The important question is that what happens on the puncture when we do the electric magnetic duality. Fortunately, this question has been extensively studied in [61], using these results, we can find a large class of 3d mirror pairs.

A. 3d Mirror for Sicilian theory

1. Rudiments of 3d $\mathcal{N}=4$ theories

3d $\mathcal{N}=4$ theories have a constrained moduli space [60]: it can have a Coulomb branch, parameterized by massless vector multiplets, and a Higgs branch, parameterized by massless hypermultiplets. There can be mixed branches as well, parameterized by both sets of fields. All branches are hyperkähler. When the theory is superconformal, it has R-symmetry $\mathrm{SO}(4)_R \simeq \mathrm{SO}(3)_X \times \mathrm{SO}(3)_Y$: then $\mathrm{SO}(3)_X$ acts on the lowest component of vector multiplets, while $\mathrm{SO}(3)_Y$ acts on that of hypermultiplets.

Both the Coulomb and Higgs branch can support the action of a global non-R symmetry group: we will call them Coulomb and Higgs symmetries respectively. When the 3d theory has a Lagrangian description, the Higgs branch is not quantum corrected and the Higgs symmetry is easily identified as the action on hypermultiplets. Coulomb symmetries are subtler. The classic example is a U(1) vector multiplet with field strength F: then $J = *F = d\phi$ is the conserved current of a U(1) Coulomb symmetry, which shifts the dual photon ϕ. Quantum corrections can enhance the Abelian Coulomb symmetry to a non-Abelian one.

To both Coulomb and Higgs symmetries are associated conserved currents. We will often use them to "gauge together" two or more theories. What we mean are the following two options. Firstly, we can take two theories—each of which has a global symmetry group G acting on the Higgs branch—then take the current of the diagonal subgroup and couple it to a G vector multiplet, in a manner which is $\mathcal{N} = 4$ and gauge invariant. Secondly, we can take two theories—each of which has a global symmetry group G acting on the Coulomb branch—and couple a G vector field to the diagonal subgroup. To preserve $\mathcal{N} = 4$ supersymmetry, one needs to use a *twisted* vector multiplet whose lowest component is non-trivially acted by $\mathrm{SO}(3)_Y$. Twisted vector multiplets can also be coupled to twisted hypermultiplets, whose lowest component is non-trivially acted by $\mathrm{SO}(3)_X$. The mirror map then relates two theories $\mathcal{A}$ and $\mathcal{B}$, such that the Coulomb branch of $\mathcal{B}$ is the Higgs branch of $\mathcal{A}$ and vice versa.

We will often consider 5d, 4d and 3d versions of a theory. What we mean is that a lower dimensional version is obtained by simple compactification on S^1. When a 4d $\mathcal{N} = 2$ theory is compactified to 3d there is a close relation between the moduli spaces of the two versions [33]. The Higgs branches are identical. If the 4d Coulomb branch has complex dimension n, the 3d Coulomb branch is a fibration of T^{2n} on the 4d Coulomb branch, and has quaternionic dimension n. The Kähler class of the torus fiber is inversely proportional to the radius of S^1. We often take the small radius limit and discuss the resulting superconformal theory.

2. Mirror of triskelions via a brane construction

The objective of this subsection is to find the mirror of the T_N theory, and more generally of triskelion theories. In the next subsection, we will explain how to gauge them together and construct the mirror of general Sicilian theories.

a. Mirror of T_N

The mirrors of a large class of theories have been found by Hanany and Witten by exploiting a brane construction [62] (see also [63, 64]): one realizes the field theory as the low energy limit of a system in IIB string theory of D3-branes suspended between NS5-branes and D5-branes. The mirror theory is obtained by performing an S-duality on the configuration, and then reading off the new gauge theory. We cannot apply this program directly to the M-theory brane construction of Sicilian theories, except for those cases that reduce to a IIA brane construction.

A 3d theory can also be studied by first constructing its 5d version using a web of 5-branes and then compactifying it on T^2 [65]. In [66] it was shown how to lift the Sicilian theories to five dimensions, and how to get a brane construction of them in IIB string theory. That paper focused on the uplift of N M5-branes wrapped on the sphere with three generic punctures, and this is all we need to start.

Consider a web of semi-infinite 5-branes in IIB string theory, made of N D5-branes, N NS5-branes and N $(1,1)$ 5-branes meeting at a point, as summarized in Figure 20. At the intersection lives a 5d theory which we call the 5d T_N theory [66], and many properties of its Coulomb branch can be read off the brane construction. Instead of keeping the 5-branes semi-infinite, we can terminate each of them at finite distance on a 7-brane of the same (p,q)-type as in Figure 20. The distance does not affect the Coulomb branch of the low energy 5d theory: a (p,q) 5-brane terminating

	0	1	2	3	4	5	6	7	8	9
D5	–	–	–	–	–	–				
NS5	–	–	–	–	–		–			
(1,1) 5-brane	–	–	–	–	–	angle				
(p,q) 7-brane	–	–	–	–	–			–	–	–

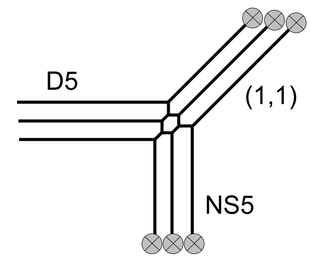


Fig. 20. Left: Table of directions spanned by the objects forming the web. Right: The web of N D5-branes, NS5-branes and $(1,1)$ 5-branes; here $N = 3$. In the figure the D5's are semi-infinite, while NS5's and $(1,1)$ 5-branes terminate each on a 7-brane $\otimes$ of the same type. The Coulomb branch of the 5d low energy theory is not sensitive to this difference.

on a (p,q) 7-brane on one side and on the web on the other side has boundary conditions that kill all massless modes [62]. However this modification is useful for three reasons: it displays the Higgs branch as normalizable deformations of the web along $x^{7,8,9}$; it admits a generalization where multiple 5-branes end on the same 7-brane (this configurations are related to generic punctures on the M5-branes, as in subsection b); upon further compactification to three dimensions, it allows us to read off the mirror theory.

Our strategy to understand the 3d T_N theory is to consider the IIB brane web on T^2, understand the low energy field theory leaving on each of the three arms separately, and finally understand how they are coupled together at the junction. We exploit the brane construction here, and present a different perspective in subsection 4.

Consider, for definiteness, the arm made of N D5-branes ending on N D7-branes. We first want to consider the arm alone, therefore we will substitute the web junction with a single D7-brane. Since the brane construction lives on T^2, we can perform two

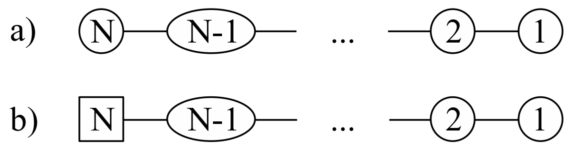


Fig. 21. a) Quiver diagram resulting from a configuration of D3-branes suspended between NS5-branes. b) Quiver diagram of the $T[\mathrm{SU}(N)]$ theory. Circles are $\mathrm{U}(r_a)$ gauge groups, the square is an $\mathrm{SU}(N)$ global symmetry group and lines are bifundamental hypermultiplets.

T-dualities and one S-duality to map it to a system of D3-branes suspended between NS5-branes—the familiar Hanany-Witten setup. We identify the SO(3) symmetry rotating $x^{7,8,9}$ with $\mathrm{SO}(3)_Y$. $\mathrm{SO}(3)_X$ will only appear in the low-energy limit, rotating the motion in the $x^{5,6}$-plane and the Wilson lines around the torus.

The low energy field theory is a linear quiver of unitary gauge groups, as in Figure 21a. Each stack of r D3-branes leads to a $\mathrm{U}(r)$ twisted vector multiplet, while each NS5-brane leads to a twisted bifundamental hypermultiplet. The other two arms made of (p, q) 5-branes and 7-branes lead to the same field theory: to read it off, we perform first an S-duality to map the system to D5-branes and D7-branes, and then proceed as above.[1]

To conclude, we need to understand what is the effect of joining the three arms together, instead of separately ending each of them on a single (p, q) 7-brane. We look at the effect on the moduli space: In each arm, the motion of the 5-branes along $x^{7,8,9}$ is parameterized by the twisted vector multiplet. When the three arms are joined together, the positions of the 5-branes at the intersection are forced to be equal, therefore the boundary condition breaks the three $\mathrm{U}(N)$ gauge groups to the

[1]The gauge couplings at intermediate energies will be different, but this will not affect the common IR fixed point to which the theories flow.

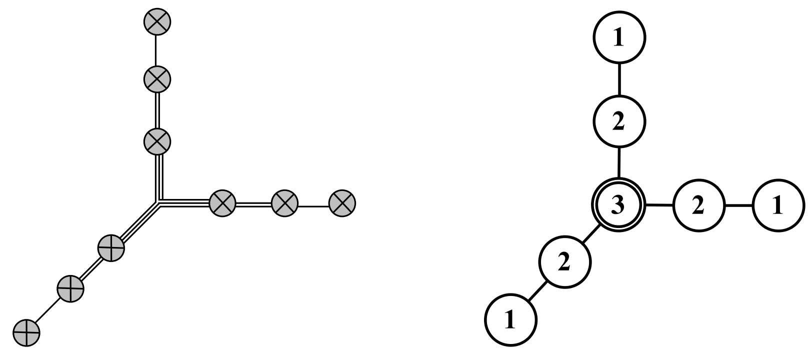


Fig. 22. Left: (p, q)-web realizing the T_N theory, with aligned 7-branes. Right: Quiver diagram of the mirror of T_N, with gauge groups $\mathrm{U}(r)$. The group at the center is taken to be SU, to remove the decoupled overall $\mathrm{U}(1)$. Here $N = 3$.

diagonal one. The resulting low energy field theory is a quiver gauge theory, depicted in Figure 22, that we will call *star-shaped*. Notice that the $\mathrm{U}(1)$ diagonal to the whole quiver is decoupled; this can be conveniently implemented by taking the gauge group at the center to be $\mathrm{SU}(N)$. Interestingly, we find that the mirror theory of T_N has a simple Lagrangian description.

In view of the subsequent generalizations, it is useful to give a slightly different but equivalent definition of the star-shaped quiver: To each maximal puncture we associate a 3d linear quiver, introduced in [61] and called $T[\mathrm{SU}(N)]$.[2] Its gauge group has the structure

$$\underline{\mathrm{SU}(N)} - \mathrm{U}(N-1) - \mathrm{U}(N-2) - \cdots - \mathrm{U}(1) \ , \tag{6.1}$$

see Figure 21b. The underlined group is a flavor symmetry, and we have bifundamental hypermultiplets between two groups. The $\mathrm{SU}(N)$ Higgs symmetry is manifest, whilst only the Cartan subgroup of the $\mathrm{SU}(N)$ Coulomb symmetry is manifest and enhancement is due to monopole operators. The star-shaped quiver is then obtained

[2]Note that this theory is distinct from the T_N theory.

by taking three $T[\mathrm{SU}(N)]$ quivers, one for each arm, and gauging together the three $\mathrm{SU}(N)$ Higgs symmetries.

b. Mirror of triskelion

We can generalize the mirror symmetry map to 3d triskelion theories. 4d triskelion theories are the low energy limit of N M5-branes wrapped on the Riemann sphere with three generic punctures. A class of half-BPS punctures is classified by Young diagrams with N boxes [11]: we will indicate them as $\rho = \{h_1, \ldots, h_J\}$ where $h_1 \geq \cdots \geq h_J$ are the heights of the columns, and J is the number of columns.

Such classification arises naturally in the IIB brane construction [66]: we allow multiple 5-branes to terminate on the same 7-brane. For each arm, the possible configurations are labeled by partitions of N, that is Young diagrams $\rho = \{h_1, \ldots, h_J\}$. In our conventions, J is the number of 7-branes and h_a is the number of 5-branes ending on the a-th 7-brane. The maximal puncture considered before is $\{1, \ldots, 1\}$. The global symmetry at each arm is easily read off as

$$G_\rho = \mathrm{S}\Big(\prod\nolimits_h \mathrm{U}(N_h) \Big) \;, \tag{6.2}$$

where N_h is the number of columns of ρ of height h, and the diagonal $\mathrm{U}(1)$ has been removed. The brane construction also makes clear that a triskelion theory with punctures (ρ_1, ρ_2, ρ_3) arises as the effective theory along the Higgs branch of T_N: it can be obtained by removing 5-branes suspended between 7-branes, and this is achieved by moving along the Higgs branch.

To construct the mirror of the 3d triskelion theory we proceed as before. We consider the three arms separately, substituting the junction with a single 7-brane. We perform an S-T^2-S duality on each arm, to map it to a system of D3-branes suspended between NS5-branes; the field theory is read off to be a 3d linear quiver

Fig. 23. Mirror theory of a generic puncture. Left: Young diagram of the puncture $\{3,2,2,1\}$. Its global symmetry is $\mathrm{U}(2)\times\mathrm{U}(1)$. Center: Corresponding configuration of 5-branes and 7-branes $\otimes$. Right: The same configuration, with the 7-branes aligned. The ranks are then read off to be 8, 5, 3, 1.

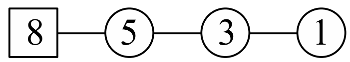


Fig. 24. $T_\rho[\mathrm{SU}(N)]$ theory, for $\rho=\{3,2,2,1\}$. All gauge groups are $\mathrm{U}(r)$.

with unitary gauge groups. These steps are summarized in Figure 23. Finally we glue together the three arms, which corresponds to setting boundary conditions that break the three $\mathrm{U}(N)$ factors to the diagonal $\mathrm{U}(N)$; the overall $\mathrm{U}(1)$ is decoupled and removed, thus making the gauge group at the center to be $\mathrm{SU}(N)$.

As before, we can construct the 3d star-shaped quiver in an equivalent way. To each puncture[3] $\rho=\{h_1,\ldots,h_J\}$ we associate a linear quiver $T_\rho[\mathrm{SU}(N)]$ [61]: it has the structure

$$\underline{\mathrm{SU}(r_0)} - \mathrm{U}(r_1) - \mathrm{U}(r_2) - \ldots - \mathrm{U}(r_{J-1}) \ , \tag{6.3}$$

where the underlined group is a flavor Higgs symmetry and the others are gauge groups. We have hypermultiplets in the bifundamental representation of $\mathrm{U}(r_i)\times\mathrm{U}(r_{i+1})$. Here r_a is given by

$$r_a = \sum_{b=a+1}^{J} h_b \ . \tag{6.4}$$

[3]We indicate both a Young diagram and the corresponding puncture with the same symbol ρ.

This quiver has $\mathrm{SU}(N)$ symmetry on the Higgs branch and (6.2) on the Coulomb branch. An example is in Figure 24. The theory $T[\mathrm{SU}(N)]$ introduced in subsection a is $T_\rho[\mathrm{SU}(N)]$ with $\rho = \{1, 1, \ldots, 1\}$, *i.e.* the maximal puncture. The 3d star-shaped quiver can be obtained by gauging together the three $\mathrm{SU}(N)$ Higgs symmetries of $T_{\rho_i}[\mathrm{SU}(N)]$ for $i = 1, 2, 3$.

Before continuing, let us recall the structure of the Higgs and Coulomb branches of this theory [61]. To a Young diagram $\rho = \{h_1, \ldots, h_J\}$, one associates a representation ρ of $\mathrm{SU}(2)$ given by

$$\rho = \underline{h_1} \oplus \underline{h_2} \oplus \cdots \oplus \underline{h_J} \tag{6.5}$$

where $\underline{h_a}$ is the irreducible h_a-dimensional representation. Let the generators of $\mathrm{SU}(2)$ be $t^\pm$ and t^3. Then $\rho(t^+) \in \mathfrak{su}(N)_\mathbb{C}$ is the direct sum of the Jordan blocks of size h_1, $\ldots$, h_J. In particular this is nilpotent. The nilpotent orbit of type ρ is defined to be

$$\mathcal{N}_\rho = \mathrm{SU}(N)_\mathbb{C} \cdot \rho(t^+)\ . \tag{6.6}$$

In particular it has an isometry $\mathrm{SU}(N)$. Its closure $\overline{\mathcal{N}_\rho}$ is a hyperkähler cone and it coincides with the Higgs branch of the quiver $T_{\rho^\mathsf{T}}[\mathrm{SU}(N)]$, where ρ^T denotes the transpose of the Young diagram ρ in which h_a are the length of the rows.

The Slodowy slice $\mathcal{S}_\rho$ is a certain nice transverse slice to $\mathcal{N}_\rho \subset \mathfrak{su}(N)_\mathbb{C}$ at $\rho(t^+)$. The Coulomb branch of $T_\rho[\mathrm{SU}(N)]$ is $\mathcal{S}_\rho \cap \overline{\mathcal{N}}$, where $\mathcal{N} = \mathcal{N}_{\{1,\ldots,1\}}$ is the maximal nilpotent orbit. Then the isometry of the Coulomb branch is the commutant of $\rho(\mathrm{SU}(2))$ inside $\mathrm{SU}(N)$, and agrees with the symmetry (6.2) read off from the brane construction.

3. Mirror of Sicilian theories

After having understood the mirror of triskelions, which are the building blocks, we can proceed to generic 3d Sicilian theories. The mirror of a 3d triskelion with punctures (ρ_1, ρ_2, ρ_3) is obtained by taking the three $T_{\rho_i}[\mathrm{SU}(N)]$ linear quivers for $i = 1, 2, 3$ and gauging together the three $\mathrm{SU}(N)$ Higgs symmetry factors. To construct a Sicilian theory we gauge together two $\mathrm{SU}(N)$ Higgs symmetries, therefore on the mirror side we gauge together two $\mathrm{SU}(N)$ Coulomb symmetries. In the following we study the effect of such gauging on the mirror.

a. Genus zero: star-shaped quivers

Let us consider, for simplicity, two triskelions glued together. The mirror is obtained by taking the two sets of linear quivers $T_{\rho_i}[\mathrm{SU}(N))]$ and $T_{\rho_i'}[\mathrm{SU}(N)]$, $i = 1, 2, 3$. We gauge together the three $\mathrm{SU}(N)$ Higgs symmetries in each set. We let ρ_1 and ρ_1' be maximal, and gauge together the $\mathrm{SU}(N)$ Coulomb symmetries of $T_{\rho_1}[\mathrm{SU}(N)]$ and $T_{\rho_1'}[\mathrm{SU}(N)]$.

Since the order of gauging does not matter, we shall first consider the effect of gauging together two copies of $T[\mathrm{SU}(N))]$ by the $\mathrm{SU}(N)$ Coulomb symmetries. The resulting low energy theory [61] has a Higgs branch $T^*\mathrm{SU}(N)_{\mathbb{C}}$, the total space of the cotangent bundle to the complexified $\mathrm{SU}(N)$ group, and no Coulomb branch. The Higgs branch is acted upon by $\mathrm{SU}(N) \times \mathrm{SU}(N)$ on the left and right respectively, but every point of the zero-section breaks it to the diagonal $\mathrm{SU}(N)$, and no other point on the moduli space preserves more symmetry. Since the Higgs branch has a scale given by the volume of the base space $\mathrm{SU}(N)_{\mathbb{C}}$ and it is smooth, around each point the theory flows to $N^2 - 1$ free twisted hypermultiplets, which are then eaten by the Higgs mechanism.

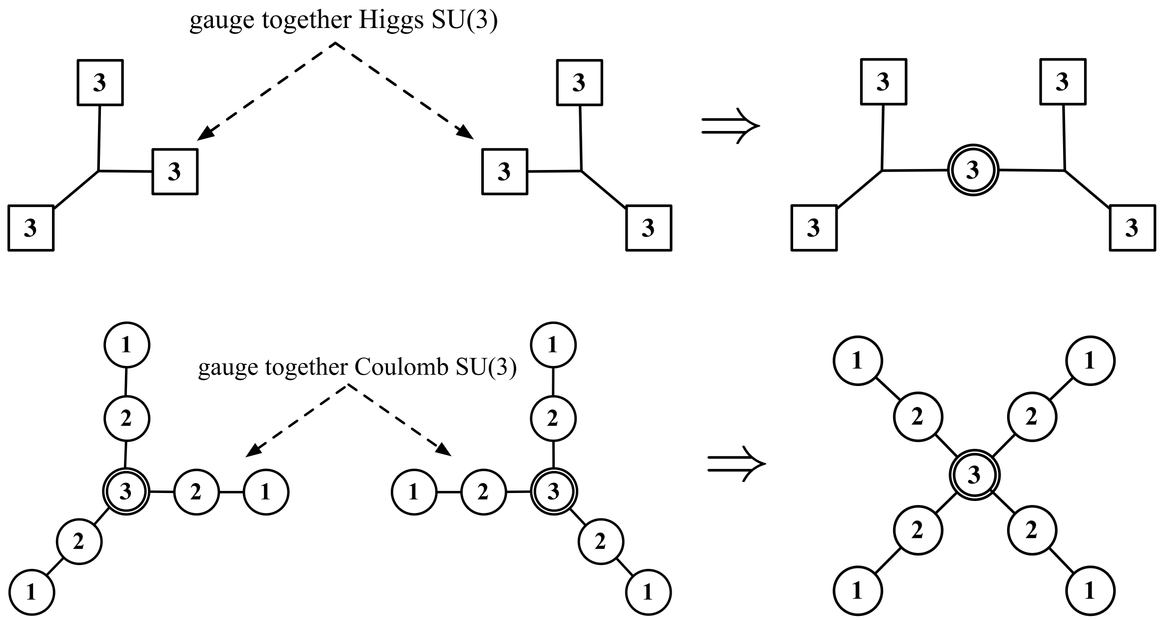
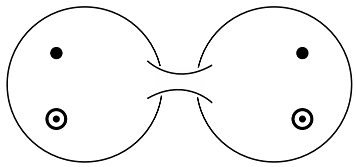


Fig. 25. Top: We take two copies of T_N and gauge together two SU(N) Higgs symmetries. Bottom: Its mirror. We gauge together two SU(N) Coulomb symmetries. This ends up eliminating the two $T[\mathrm{SU}(N)]$ tails. Here $N = 3$.

Summarizing, coupling two copies of $T[\mathrm{SU}(N)]$ by their SU(N) Coulomb symmetries spontaneously breaks the SU(N) $\times$ SU(N) Higgs symmetry to the diagonal subgroup. Therefore, we are left with $T_{\rho_{2,3}}[\mathrm{SU}(N)]$ and $T_{\rho'_{2,3}}[\mathrm{SU}(N)]$ with all four SU(N) Higgs symmetries gauged together. See Figures 25 and 26 for examples; there, the T_N theory is depicted by a trivalent vertex with three boxes, each representing an SU(N) Higgs symmetry.

This is easily generalized to a generic 3d genus zero Sicilian theory obtained from a sphere with punctures. Its mirror is obtained by taking the set of $T_\rho[\mathrm{SU}(N)]$ linear quivers corresponding to all punctures, and gauging all the SU(N) Higgs symmetries together. Such theory is a star-shaped quiver.

We find that in 3d, the low energy theory only depends on the topology of the punctured Riemann surface, and not on its complex structure. This is as expected:



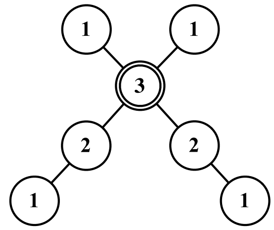


Fig. 26. Left: A pants decomposition of the Riemann surface of a genus zero Sicilian theory. In the example the four punctures are $\{1,1,1\}$ ($\odot$) and $\{2,1\}$ ($\bullet$), and the Sicilian theory is SU(3) SQCD with 6 flavors. Right: Star-shaped quiver, mirror of a genus zero Sicilian theory. Notice that the $\mathrm{SU}(6)\times\mathrm{U}(1)$ Coulomb symmetry due to monopole operators is easy to see.

In 4d the complex structure controls the complexified gauge couplings of the IR fixed point. When compactifying to 3d, all gauge couplings flow to infinity based on dimensional analysis, washing out the information contained therein.

b. Higher genus: adjoint hypermultiplets

Let us next consider the mirror of 3d Sicilian theories obtained from Riemann surfaces of genus $g \geq 1$. Taking advantage of S-duality in 4d Sicilian theories, without loss of generality we can consider a pants decomposition in which all handles come from gluing together two maximal punctures on the same triskelion.

The mirror can be constructed as before, by taking $T_\rho[\mathrm{SU}(N)]$ for each of the punctures, and suitably gauging together the Higgs and Coulomb $\mathrm{SU}(N)$ symmetries. The only difference compared to the genus zero case is that, for each of the g handles, we get two copies of $T[\mathrm{SU}(N)]$ gauged together both on the Higgs and Coulomb branch. This amounts to gauging the diagonal subgroup of the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ Higgs symmetry of $T^*\mathrm{SU}(N)_{\mathbb{C}}$, which is not broken along the zero-section: the N^2-1 twisted hypermultiplets living there are thus left massless. They transform in the

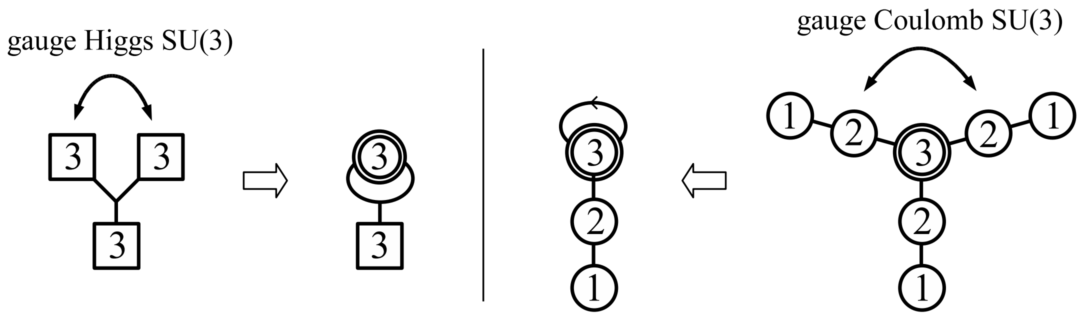


Fig. 27. Mirror symmetry of higher genus Sicilian theories. Left: We gauge two SU(N) Higgs symmetries producing a "handle" in a higher genus Sicilian theory. Right: In the mirror we gauge two SU(N) Coulomb symmetries, getting rid of two $T[\mathrm{SU}(N)]$ tails but leaving one adjoint hypermultiplet. Here $N = 3$.

adjoint representation of the diagonal subgroup. See Figure 27 for an example.

We find that the mirror theory is, as before, a gauge theory. It is obtained by taking the set of $T_\rho[SU(N)]$ linear quivers corresponding to the punctures, plus g twisted hypermultiplets in the adjoint representation of SU(N), and gauging all the SU(N) Higgs symmetries together. See Figure 28 for two examples and the generic case. The g adjoint hypermultiplets carry an accidental IR USp($2g$) Higgs symmetry, not present in the 4d theory. Again, the 3d IR fixed point only depends on the topology of the defining Riemann surface.

One can check that the dimensions of the Coulomb and Higgs branch in the 3d Sicilian theories and star-shaped quivers agree, after exchange.

Let us stress two nice examples of mirror symmetry. One is the 4d theory dual to the Maldacena-Nuñez supergravity solution [67] of genus g. This theory in non-Lagrangian, however after compactification to three dimensions its mirror is SU(N) with g adjoint hypermultiplets (center in Figure 28). The other example is the rank-k $E_{6,7,8}$ theories. Their 3d mirror is a quiver of groups U($k\, n_i$) where the shape is the

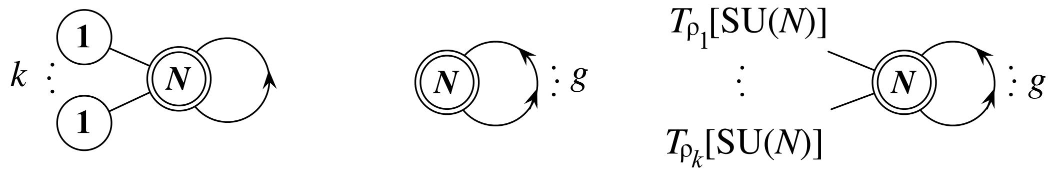


Fig. 28. Star-shaped mirrors of genus g Sicilian theories. Left: Mirror of the Sicilian theory of genus 1 with k simple punctures $\{N-1,1\}$. The Sicilian theory is a closed chain quiver (also called elliptic quiver) of k SU(N) gauge groups and bifundamentals. Center: Mirror of the Sicilian theory of genus g and without punctures, which is the 3d compactification of the field theory dual to the Maldacena-Nuñez supergravity solution. Right: Mirror of a generic genus g Sicilian theory with k punctures $\rho_1, \ldots, \rho_k$. The g hypermultiplets are in the adjoint of SU(N), and a USp($2g$) IR symmetry emerges.

extended Dynkin diagram $\hat{E}_{6,7,8}$ and the ranks are k times the Dynkin index n_i of the i-th node. For $k = 1$ it is the example considered in the seminal paper [60].

4. Boundary conditions, mirror symmetry and $\mathcal{N} = 4$ SYM on a graph

In the last subsection we described how to obtain the mirror of 3d Sicilian theories in terms of junctions of 5-branes compactified on T^2. Since a stack of N 5-branes compactified on T^2 gives $\mathcal{N} = 4$ super Yang-Mills, it is possible to rephrase what we derived from the brane construction in terms of half-BPS boundary conditions of $\mathcal{N} = 4$ super Yang-Mills, as was in [61]. This perspective allows us to extend the mirror symmetry map to more general theories, not easily engineered with M5-branes. Let us start by reviewing the framework of [68, 61].

a. Half-BPS boundary conditions: review

Consider $\mathcal{N} = 4$ super Yang-Mills with gauge group $G = G_1 \times G_2 \times \cdots$ on a half-space $x^3 > 0$. In the following we set all θ angles to zero. We introduce the metric on the

Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots$ using the coupling constants, as in

$$\langle a, b\rangle_{\mathfrak{g}} = g_1^{-2}\langle a_1, b_1\rangle_{\mathfrak{g}_1} + g_2^{-2}\langle a_2, b_2\rangle_{\mathfrak{g}_2} + \cdots \tag{6.7}$$

where $a = a_1 \oplus a_2 \oplus \cdots$ and $b = b_1 \oplus b_2 \oplus \cdots$ are two elements of $\mathfrak{g}$, $\langle \cdots \rangle_{\mathfrak{g}_i}$ is the standard Killing metric on $\mathfrak{g}_i$, and g_i is the coupling constant of the i-th factor. The Lagrangian is then given by

$$S = \int d^4x \langle F_{\mu\nu}, F^{\mu\nu}\rangle + \langle D_\mu \Phi_i, D^\mu \Phi^i\rangle + \text{fermions} . \tag{6.8}$$

We split the six adjoint scalar fields $\Phi_{1,\ldots,6}$ into $\vec{X} = (X_1, X_2, X_3)$ and $\vec{Y} = (Y_1, Y_2, Y_3)$. Out of the SU(4) R-symmetry, the symmetry manifest under this decomposition is the subgroup $\mathrm{SO}(3)_X \times \mathrm{SO}(3)_Y$, which we can identify with the $\mathrm{SO}(4)_R$ symmetry of a 3d $\mathcal{N} = 4$ CFT, as was discussed in subsection 1.

The boundary condition studied in [68] consists of the data $(\rho, H, \mathcal{B})$. First, ρ is an embedding

$$\rho : \mathrm{SU}(2) \to G \tag{6.9}$$

which controls the divergence of $\vec{X}$:

$$X_i \sim \frac{\rho(t_i)}{x^3} , \tag{6.10}$$

where t_i $(i = 1, 2, 3)$ are three generators of SU(2). The gauge field close to $x^3 = 0$ needs to commute with $\rho(\mathrm{SU}(2)) \subset G$. Therefore let H be a subgroup of G that commutes with $\rho(\mathrm{SU}(2))$, and $\mathcal{B}$ be a 3d $\mathcal{N} = 4$ CFT living on the boundary with H global symmetry. The theory $\mathcal{B}$ can possibly be an empty theory, $\varnothing$. The boundary conditions we impose are

$$0 = F^{+}_{3a}| \,, \qquad\qquad 0 = F^{-}_{ab}| \,, \tag{6.11a}$$

$$0 = \vec{X}^{+} + \vec{\mu}_{\mathcal{B}}| \,, \qquad\qquad 0 = D_3\vec{X}^{-}| \,, \tag{6.11b}$$

$$0 = D_3\vec{Y}^{+}| \,, \qquad\qquad 0 = \vec{Y}^{-}| \,. \tag{6.11c}$$

Here the indices $a, b = 0, 1, 2$ are the directions along the boundary, and the bar $|$ means the value at the boundary $x^3 = 0$. We decompose the algebra of G as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Then the superscript $^+$ is the projection onto $\mathfrak{h}$ and the superscript $^-$ the projection onto $\mathfrak{h}^{\perp}$. Finally $\vec{\mu}_{\mathcal{B}}$ is the moment map of the H symmetry on the Higgs branch of $\mathcal{B}$. The condition (6.11a) means that on the boundary only the gauge field in H is non-zero; in other words the boundary condition sets Neumann boundary conditions in the subalgebra $\mathfrak{h}$ and Dirichlet boundary conditions in the orthogonal complement $\mathfrak{h}^{\perp}$. Note that this class of boundary conditions does *not* treat $\vec{X}$ and $\vec{Y}$ equally: we will denote the boundary conditions more precisely as $(\rho, H, \mathcal{B})_{X,Y}$ when necessary.

We can define a boundary condition $(\rho', H', \mathcal{B}')_{Y,X}$ where the role of $\vec{X}$ and $\vec{Y}$ is interchanged; in particular we will have $\vec{Y}^{+} + \vec{\mu}_{\mathcal{B}'} = 0$, where $\vec{\mu}_{\mathcal{B}'}$ is the moment map of H' on the Higgs branch of the twisted hypermultiplets of $\mathcal{B}'$. The S-duality of $\mathcal{N} = 4$ SYM in the bulk $x^3 > 0$ acts non-trivially on spinors, and it is known to map the class of boundary conditions $(\rho, H, \mathcal{B})_{X,Y}$ to another one with the role of $\vec{X}$ and $\vec{Y}$ exchanged:

$$S : (\rho, H, \mathcal{B})_{X,Y} \mapsto (\rho', H', \mathcal{B}')_{Y,X} \,. \tag{6.12}$$

Let us emphasize again that this involves the exchange of the role of untwisted and twisted multiplets of the boundary 3d theory, and it is closely related to mirror symmetry.

		0	1	2	3	4	5	6	7	8	9	10
IIB	D3	–	–	–	–		□					
	D5	–	–	–			□	–	–	–		
IIA	D4	–	–	–	–	–	□					
	D6	–	–	–		–	□	–	–	–		
M	M5	–	–	–	–	–	–					
	KK	–	–	–		–		–	–	–		
IIA	D4	–	–	–	–	□	–					
	KK	–	–	–		□		–	–	–		
IIB	D3	–	–	–	–	□						
	NS5	–	–	–		□	–	–	–	–		

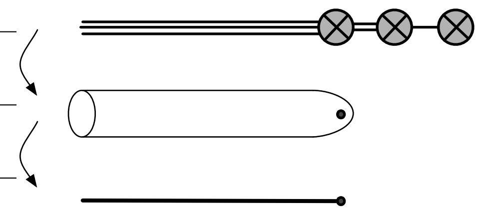


Fig. 29. Boundary conditions of 4d $\mathcal{N} = 4$ SYM from 5d $\mathcal{N} = 2$ SYM. By T-duality, an arm of 5-branes ending on 7-branes (top of figure) is mapped to D3/D5 or D4/D6 (first two lines of table). Uplift to M-theory gives M5-branes on a punctured cigar—a complex surface in the KK monopole—and further reduction gives D4-branes on the same cigar (middle of figure). Reduction along the S^1 of the cigar gives 4d $\mathcal{N} = 4$ SYM on a half-space $x^3 \geq 0$ with half-BPS boundary conditions (bottom of figure). In the table we perform T-duality and uplift/reduction along $x^{4,5}$. A square means coordinates to be removed.

For $G = \mathrm{SU}(N)$, it was shown in [61] that

$$S : \ \big(1,\, \mathrm{SU}(N),\, T_\rho[\mathrm{SU}(N)]\big)_{X,Y} \ \mapsto \ \big(\rho,\, 1,\, \varnothing\big)_{Y,X} \,. \tag{6.13}$$

For example, $\rho = \{1, \ldots, 1\}$, which we abbreviate as just $\rho = 1$, is the trivial embedding and $(1, 1, \varnothing)$ is the standard Dirichlet boundary condition, which can be realized by ending N D3's on N D5-branes. Its S-dual has $T[\mathrm{SU}(N)]$ on the boundary, and comes from ending N D3's on N NS5-branes. On the other extreme, the theory $T_{\{N\}}[\mathrm{SU}(N)]$ is an empty theory and $\big(1, \mathrm{SU}(N), \varnothing\big)$ is the Neumann boundary condition. This can be realized by ending N D3's on 1 NS5-brane (and decoupling the U(1)). Its S-dual is $(\{N\}, 1, \varnothing)$, and corresponds to ending N D3's on 1 D5-brane.

This pair of boundary conditions arise naturally from 5-branes ending on 7-branes, see Figure 29. Start from D5-branes ending on D7-branes. Compactification on T^2 and T-duality leads to a configuration of N D3-branes ending on D5-branes. This realizes the boundary condition $(\rho, 1, \varnothing)$ of 4d $\mathcal{N} = 4$ SYM , on the right of (6.13). When only one T-duality is performed, it can also be thought of as N M5-branes on a cylinder ending on a cap with a puncture inserted, or a punctured cigar, further compactified on S^1. Then it can be thought of as N D4-branes on the same cigar geometry. The Kaluza-Klein reduction along the S^1 of the cigar produces 4d SYM on a half-space, and since the original system preserves half of the supersymmetry, the boundary condition is also half-BPS. In fact, this corresponds to N D3-branes ending on NS5-branes, and realizes the boundary condition $(1, \mathrm{SU}(N), T_\rho[\mathrm{SU}(N)])$ of 4d SYM on the left of (6.13): we have just performed S-duality.

b. Junction and boundary conditions from the brane web

In our brane construction, N (p, q) 5-branes on the torus give $\mathcal{N} = 4$ $\mathrm{U}(N)$ super Yang-Mills. The 6d gauge coupling of a (p, q) 5-brane is inversely proportional to its tension. Compactifying on T^2 and performing S-duality of the resulting 4d theory, its action is given schematically by

$$T \int d^4x \left(\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \mathrm{tr}\, \partial_\mu X_i \partial^\mu X_i + \mathrm{tr}\, \partial_\mu Y_i \partial^\mu Y_i \right) . \tag{6.14}$$

Here T is the tension of the 5-brane multiplied by the area of T^2, $Y_{1,2,3}$ is a fluctuation along $x^{7,8,9}$, X_1 is the fluctuation transverse to the brane inside $x^{5,6}$ and $X_{2,3}$ come from the Wilson lines around T^2.

We would like to understand the boundary condition corresponding to the junction of N D5-, NS5- and $(1, 1)$ 5-branes, see Figure 30. Let us first consider the case $N = 1$. Let us denote the unit normal to the 5-branes by $\vec{n}_{1,2,3}$ and the tensions of

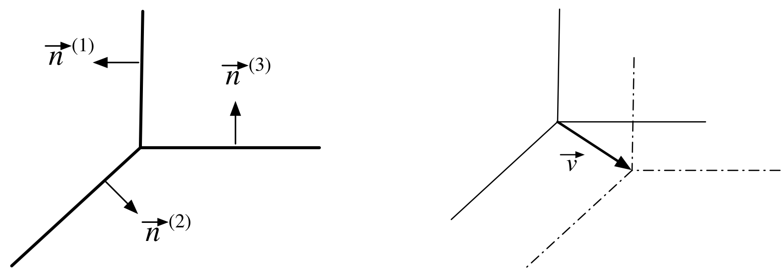


Fig. 30. Left: In the $x^{5,6}$-plane, we measure the worldvolume displacements $X_1^{(1,2,3)}$ of the three segments along the unit normal vectors $\vec{n}_{(1,2,3)}$. The $\vec{Y}^{(i)}$ displacements are along $x^{7,8,9}$, "orthogonal" to the paper. Right: When the junction is moved by $\vec{v}$, the displacements are given by $X_1^{(i)} = \vec{v} \cdot \vec{n}^{(i)}$.

the 5-branes by $T_{1,2,3}$. The condition of the balance of forces can be written as

$$T_1\vec{n}_1 + T_2\vec{n}_2 + T_3\vec{n}_3 = 0 \; . \tag{6.15}$$

The three arms provide three copies of U(1) SYM, that we can think of as a single U(1)^3 SYM. We measure $X_1^{(1,2,3)}$ along the normal $\vec{n}^{(1,2,3)}$ of each of the 5-branes. The boundary condition for the scalar X_1 is $T_1X_1^{(1)} + T_2X_1^{(2)} + T_3X_1^{(3)} = 0$ that we expect to be enhanced to

$$T_1\vec{X}^{(1)} + T_2\vec{X}^{(2)} + T_3\vec{X}^{(3)} = 0 \tag{6.16}$$

after compactification on T^2. On the other hand, the boundary condition for $\vec{Y}$ is just

$$\vec{Y}^{(1)} = \vec{Y}^{(2)} = \vec{Y}^{(3)} \tag{6.17}$$

because they can only move along $x^{7,8,9}$ together. Comparing with the formulation in (6.11), these boundary conditions can be expressed in the two equivalent, S-dual ways

$$S: \; (1,\, \mathrm{U}(1)_{\mathrm{diag}},\, \varnothing)_{X,Y} \quad \mapsto \quad (1,\, \mathrm{U}(1)^3/\mathrm{U}(1)_{\mathrm{diag}},\, \varnothing)_{Y,X} \; , \tag{6.18}$$

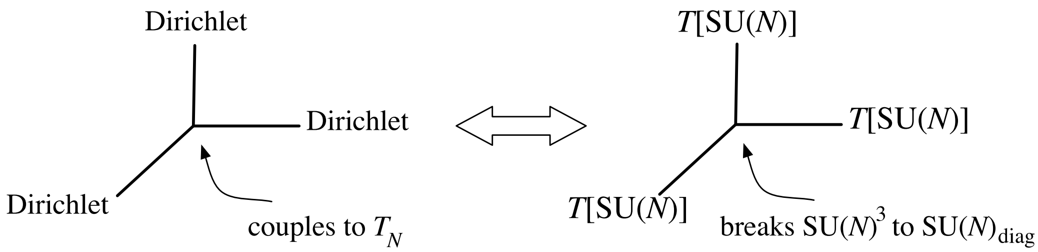


Fig. 31. Left: $\mathcal{N} = 4$ SU(N) SYM on the segments, boundary conditions $(1, \mathrm{SU}(N)^3, T_N)$ at the center and $(1, 1, \varnothing)$ at the punctures. Right: Its S-dual; boundary conditions $(1, \mathrm{SU}(N)_{\mathrm{diag}}, \varnothing)$ at the center and $(1, \mathrm{SU}(N), T[\mathrm{SU}(N)])$ at the punctures.

where $\mathrm{U}(1)_{\mathrm{diag}}$ is the diagonal subgroup of $\mathrm{U}(1)^3$. Note that the relation (6.16) determines the orthogonal complement to the diagonal subgroup under the metric of $\mathrm{U}(1)^3$ (6.7) given by the coupling constants.

The S-duality/mirror symmetry of the two conditions in (6.18) is easily checked. Consider the expression on the left, and close each arm at the external end with Neumann boundary conditions $(1, \mathrm{U}(1), \varnothing)_{X,Y}$: this gives one 3d free vector multiplet. Now consider the expression on the right and close each arm with the S-dual boundary conditions, namely Dirichlet $(1, 1, \varnothing)_{Y,X}$: this gives one free twisted hypermultiplet.

For generic N we still have the conditions (6.16)–(6.17), as can be checked by separating the N simple junctions along the $\vec{Y}$ direction. The decoupled overall U(1) part is the same as before. Then the boundary condition for the SU(N) part is

$$(1,\, \mathrm{SU}(N)_{\mathrm{diag}},\, \varnothing)_{X,Y}\ . \tag{6.19}$$

To obtain its S-dual boundary conditions, we proceed as follows. We take a trivalent graph with SU(N) $\mathcal{N} = 4$ SYM on each arm, boundary conditions $\big(1, \mathrm{SU}(N), T[\mathrm{SU}(N)]\big)$ at the external end of each arm, and the breaking-to-the-diagonal boundary condition

$(1, \mathrm{SU}(N)_{\mathrm{diag}}, \varnothing)$ at the junction (see right panel in Figure 31). This configuration realizes, at low energy, the quiver diagram in Figure 22. As found in subsection a, this quiver is the mirror of the T_N theory. On the other hand, we can directly perform S-duality on the configuration of SYM on a graph: on each arm we still have $\mathrm{SU}(N)$ SYM (which is self-dual), at the external end of each arm we get $(1, 1, \varnothing)$, while at the junction we get the boundary condition we are after (see left panel in Figure 31). Since $(1, 1, \varnothing)$ is the usual Dirichlet boundary condition, to obtain T_N which has $\mathrm{SU}(N)^3$ Higgs symmetry it must be

$$S: \; \big(1,\, \mathrm{SU}(N)_{\mathrm{diag}},\, \varnothing\big)_{X,Y} \;\mapsto\; \big(1,\, \mathrm{SU}(N)^3,\, T_N\big)_{Y,X} \,. \tag{6.20}$$

This can be proved also by considering a simple case in which we already know the mirror symmetry map. For instance, consider the 3d Sicilian theory given by one puncture $\rho = \{N\}$ on the torus: this is 3d $\mathcal{N} = 8$ $\mathrm{SU}(N)$ SYM. The graph construction has two $\mathrm{SU}(N)$ segments, $(1, \mathrm{SU}(N)^3, T_N)$ at the junction and Dirichlet boundary condition at the puncture. The mirror theory is $\mathcal{N} = 8$ $\mathrm{SU}(N)$ SYM itself. The S-dual graph has $\mathrm{SU}(N)$ on the segments and Neumann boundary condition at the puncture. To reproduce the mirror, we need $(1, \mathrm{SU}(N)_{\mathrm{diag}}, \varnothing)$ at the junction.

c. Junction and boundary conditions from 5d SYM

We can derive the boundary condition of diagonal breaking (6.19) also from 5d SYM on the punctured Riemann surface. The 3d T_N theory arises from N D4-branes on a three-punctured sphere $\mathcal{C}$. At low energy we get maximally-supersymmetric 5d SYM on $\mathcal{C}$, which has $\mathrm{U}(N)$ gauge field A_μ, curvature $F_{\mu\nu}$ and scalar fields $X_{1,2}$, $Y_{1,2,3}$. To preserve supersymmetry, the theory is twisted so that $X_{1,2}$ are effectively one-forms

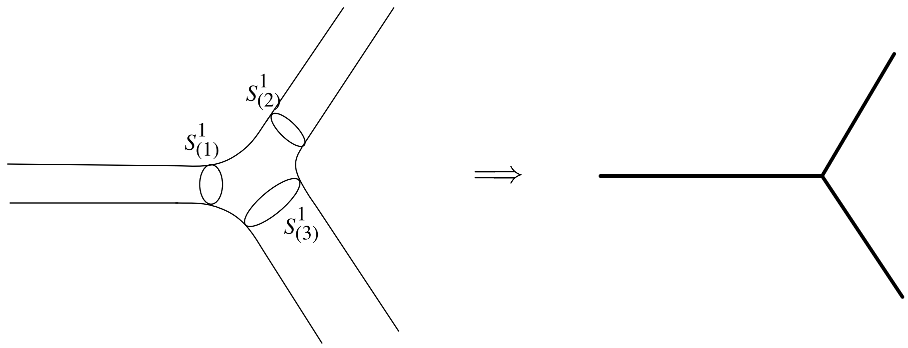


Fig. 32. Left: 5d $\mathcal{N} = 2$ SYM on a junction of three cylinders. Right: The KK reduction leads to 4d $\mathcal{N} = 4$ SYM on three half-spaces, meeting on $\mathbb{R}^3$.

on $\mathcal{C}$. The action of the bosonic sector is roughly given by

$$\frac{1}{g_{5d}^2}\int d^5x \left[\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \mathrm{tr}\, D_{[\mu}X_{\nu],a}D^{[\mu}X_a^{\nu]} + \mathrm{tr}\, D_\mu Y_a D^\mu Y_a\right] \tag{6.21}$$

where D is the covariant derivative.

We can introduce a metric on $\mathcal{C}$ such that the surface consists of three cylinders of circumference $\ell_{1,2,3}$ meeting smoothly at a junction, see Figure 32. The behavior of the system at length scales L far larger than $\ell_{1,2,3}$ is given by three segments of 4d $\mathcal{N} = 4$ SYM meeting at the same boundary $\mathbb{R}^3$. The action on each segment is (6.14) with $T_i = \ell_i/g_{5d}^2$. The boundary condition at the junction is half-BPS, because the original 5d SYM on $\mathcal{C}$ is half-BPS.

Let us determine the boundary condition explicitly. Classical configurations which contributes dominantly to the path integral at the scale $L \gg \ell_{1,2,3}$ will have A, X, Y of order L^{-1} and the action density should scale as L^{-4}. Mark three $S^1_{(i)}$'s ($i = 1, 2, 3$) very close to the junction as depicted in Figure 32, and call the region bounded by them as S. When L is very big, the non-liner term in the covariant derivative can be discarded compared to the derivative, and the dominant contribution to the action is

$$\sim \int_S d^5x \left[|dA|^2 + |dX_a|^2 + |\partial Y_a|^2\right] \ . \tag{6.22}$$

The action density should be of order L^{-4}. Then in the large L limit, Y_a need to be constant while A and X_a need to be flat. We let $A^{(i)}$, $X_{1,2}^{(i)}$ and $Y_{1,2,3}^{(i)}$ be the values of A, $X_{1,2}$ and $Y_{1,2,3}$ on $S^1_{(i)}$. The boundary condition for Y is then given by

$$\vec{Y}^{(1)} = \vec{Y}^{(2)} = \vec{Y}^{(3)} \ . \tag{6.23}$$

Flatness of A translates to

$$\int_{S^1_{(1)}} A + \int_{S^1_{(2)}} A + \int_{S^1_{(3)}} A = 0 \tag{6.24}$$

giving $T_1A^{(1)} + T_2A^{(2)} + T_3A^{(3)} = 0$ and similarly for $X_{1,2}$. Calling A as X_3, we obtain

$$T_1\vec{X}^{(1)} + T_2\vec{X}^{(2)} + T_3\vec{X}^{(3)} = 0 \ . \tag{6.25}$$

The result agrees with what we deduced from the brane construction in (6.16) and (6.17). However the derivation here has the merit that it is applicable also to the 6d $\mathcal{N} = (2,0)$ theories of type D and E, for which we have not found a brane construction of the junction.

d. $\mathcal{N} = 4$ SYM on a graph

We found that 3d Sicilian theories can be engineered in a purely field theoretic way—without involving string theory anymore—by putting $\mathrm{SU}(N)$ $\mathcal{N} = 4$ SYM on a graph. The graph is made of segments, that can end on "punctures" or can be joined at trivalent vertices. On each segment we put a copy of $\mathrm{SU}(N)$ SYM. A puncture ρ corresponds to the boundary condition $(\rho, 1, \varnothing)$, while the trivalent vertex corresponds to the boundary condition $(1, \mathrm{SU}(N)^3, T_N)$.

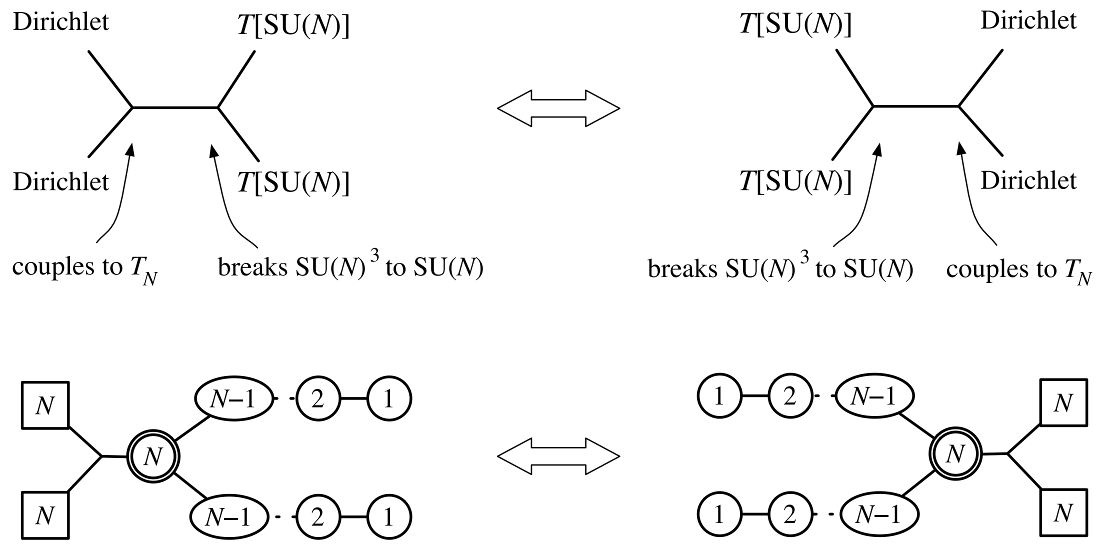


Fig. 33. Mirror symmetry including both T_N and star-quivers. The 3d theory is given by a graph on which $\mathcal{N}=4$ SYM lives. The mirror is obtained by performing the S-dual of the boundary conditions at open ends and at the junctions. The case depicted is self-mirror.

To obtain the mirror theory we simply perform S-duality of $\mathcal{N}=4$ SYM on each segment: $\mathrm{SU}(N)$ SYM is mapped to itself; the boundary conditions at the punctures are mapped to $\big(1, \mathrm{SU}(N), T_\rho[\mathrm{SU}(N)]\big)$; the boundary condition at the vertices is mapped to $(1, \mathrm{SU}(N)_{\mathrm{diag}}, \varnothing)$. To read off the 3d theory it is convenient to reduce the graph: every time we have SYM with breaking-to-the-diagonal vertices on both sides, the gauge group is broken, we can remove the segment and leave a n-valent vertex which breaks $\mathrm{SU}(N)^n$ to the diagonal $\mathrm{SU}(N)$. If instead the two ends of the same segment are joined together, we are left with an adjoint hypermultiplet. This parallels the discussion of subsection 3 and reproduces the star-shaped quivers.

The advantage of this perspective is that, being purely field theoretical, can be generalized beyond brane constructions. For instance, we could couple star-shaped quivers to Sicilian theories: in this way we get a class of theories closed under mirror

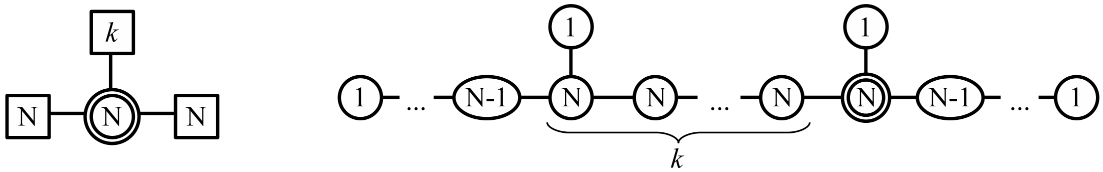


Fig. 34. Left: k domain walls introducing one extra fundamental have been added, compared to Figure 26. Right: Its mirror have k domain walls, each introducing an extra bifundamental coupled to a U(1).

symmetry, see Figure 33. More generally, the full set of half-BPS boundary conditions in [68, 61] can be used. One example is a domain wall that introduces a fundamental hypermultiplet; its mirror is a domain wall that introduces a bifundamental coupled to extra U(1), see Figure 34. Finally, one could consider $\mathcal{N} = 4$ SYM with gauge groups other than SU(N). We will consider SO(P) in the next subsection.

5. D_N Sicilian theories

Another class of Sicilian theories, that we will call of type D_N and studied in [44], can be obtained by compactification of the 6d $\mathcal{N} = (2, 0)$ D_N theory on a Riemann surface with half-BPS punctures. The 6d D_N theory is the low energy theory on a stack of $2N$ $\frac{1}{2}$M5-branes on top of the $\mathbb{R}^5/\mathbb{Z}_2$ orientifold in M-theory; here and in the following, having $2N$ $\frac{1}{2}$branes means to have $2N$ branes on the covering space. This parallels the construction of Sicilian theories of type A_{N-1} considered so far. We are interested in extending the mirror map to those theories.

Since the 6d D_N theory compactified on T^2 gives 4d $\mathcal{N} = 4$ SO($2N$) SYM at low energy, it should be possible to construct 3d D_N Sicilian theories through $\mathcal{N} = 4$ SYM on a graph, with suitable half-BPS boundary conditions at the punctures and at the junctions. This is the approach we follow in this subsection.

6. The punctures

Let us start by focusing on a single puncture, which can be understood via systems of D4/O4/D6-branes. First consider the 6d D_N theory on a cigar, with a single puncture at the tip, as we did for the A_{N-1} theory in Figure 29. Far from the tip we have the 6d D_N theory on S^1, in other words $2N$ $\frac{1}{2}$D4-branes on top of an O4$^-$-plane. The tip of the cigar with a puncture then becomes a half-BPS boundary condition for that theory, which comes from terminating $\frac{1}{2}$D4-branes on $\frac{1}{2}$D6-branes. The configuration of branes is as in the case of A_{N-1}, see the table in Figure 29. In our conventions USp(2) $\cong$ SU(2).

Let us classify how $2N$ $\frac{1}{2}$D4-branes on top of an O4$^-$-plane can end on $\frac{1}{2}$D6-branes. Let us put as many D6-branes as possible away from the orientifold. For each $\frac{1}{2}$D6, assign a column of boxes whose height is given by the change in the D4-charge across the D6-brane. We thus obtain Young diagrams with $2N$ boxes. Let N_h be the number of columns of height h. When N_h is even, we can place the N_h $\frac{1}{2}$D6-branes outside the O-plane, and no further restrictions apply. When N_h is odd, one $\frac{1}{2}$D6 has to be placed on top of the O-plane. However, every time a $\frac{1}{2}$D6 crosses the O4$^-$, the latter becomes an $\widetilde{\text{O4}}^-$ on the other side, see [69] for more details. Therefore the difference of the $\frac{1}{2}$D4-charge is odd. This implies that N_h must be even for h even. We call these the *positive* punctures, and the corresponding diagrams Young diagrams of $\mathrm{O}(2N)$. The global symmetry algebra at these punctures is read off from the brane construction:

$$\mathfrak{g}_{\rho^+} = \bigoplus_{h \text{ odd}} \mathfrak{so}(N_h) \oplus \bigoplus_{h \text{ even}} \mathfrak{usp}(N_h)\ . \tag{6.26}$$

We also have *negative* punctures, which produce a branch cut or twist line across which there is a $\mathbb{Z}_2$ monodromy of the D_N theory.[4] The monodromy will terminate on some other negative puncture on the Riemann surface. Compactifying the 6d D_N theory on S^1 with such a twist, we obtain 5d $\mathcal{N}=2$ USp$(2N-2)$ SYM [46]. This time we have $(2N-2)$ $\frac{1}{2}$D4-branes on top of an O4$^+$-plane. The property of O4$^+$-planes crossing a $\frac{1}{2}$D6-brane now implies that N_h must be even when h is odd, in contrast to the positive punctures. We call these diagrams Young diagrams of USp$(2N-2)$. The global symmetry is now

$$\mathfrak{g}_{\rho^-} = \bigoplus_{h \text{ even}} \mathfrak{so}(N_h) \oplus \bigoplus_{h \text{ odd}} \mathfrak{usp}(N_h) \ . \tag{6.27}$$

The analysis here is equivalent to that given in [44], except that we moved all the D6-branes to the far-right of the NS5-branes and that we can thus read off the flavor symmetry. So far we have considered the 6d D_N theory on a cigar, which provides information about the 4d Sicilian theory; after compactification on S^1 we can perform a T-duality and repeat the whole construction in terms of D3/O3/D5-branes, which is useful to get the mirror.

The S-dual of the boundary conditions at the punctures are easily obtained from the brane construction, as written in [61]. We start with the brane setup of the puncture, given by $\frac{1}{2}$D3-branes on top of an O3-plane and ending on $\frac{1}{2}$D5-branes, and perform an S-duality transformation (Table V). The resulting theory at the puncture is read off, recalling that $2k$ $\frac{1}{2}$D3-branes on O3$^+$ or $\widetilde{\text{O3}}^+$ and suspended between $\frac{1}{2}$NS5-branes give an USp$(2k)$ gauge theory, while k $\frac{1}{2}$D3-branes on O3$^-$ or $\widetilde{\text{O3}}^-$ give an O(k) gauge theory.[5]

[4]The 6d D_N theory on a Riemann surface has operators of spin $2, 4, \ldots, 2N-2$ plus one operator of spin N. They correspond to the Casimirs of $\mathfrak{so}(2N)$, the last one being the Pfaffian. The $\mathbb{Z}_2$ twist changes the sign of the operator of spin N, corresponding to the parity outer automorphism of $\mathfrak{so}(2N)$.

[5]At the level of the algebra, O3$^-$ and $\widetilde{\text{O3}}^-$ project $\mathfrak{u}(k)$ to its imaginary subalgebra

O-plane	gauge theory	across $\frac{1}{2}$D$(p+2)$	across $\frac{1}{2}$NS5	S-dual $(p=3)$
Op^-	O$(2N)$	$\widetilde{\mathrm{O}p}^-$	Op^+	O3^-
$\widetilde{\mathrm{O}p}^-$	O$(2N+1)$	Op^-	$\widetilde{\mathrm{O}p}^+$	O3^+
Op^+	USp$(2N)$	$\widetilde{\mathrm{O}p}^+$	Op^-	$\widetilde{\mathrm{O}3}^-$
$\widetilde{\mathrm{O}p}^+$	USp$'(2N)$	Op^+	$\widetilde{\mathrm{O}p}^-$	$\widetilde{\mathrm{O}3}^+$

Table V. Properties of Op-planes, for $p \leq 5$. We indicate: type of Op-plane, gauge theory living on them when $2N$ Dp-branes are added to the covering space, type of Op-plane on the other side of a crossing $\frac{1}{2}$D$(p+2)$-brane, or $\frac{1}{2}$NS5-brane, and S-dual plane (for $p=3$). In our conventions USp$(2) \cong$ SU(2), and USp$'(2N)$ is USp$(2N)$ with the θ angle shifted by π.

A positive puncture $\rho^+ = \{h_1, \ldots, h_J\}$ before S-duality describes D3-branes on an O3^- puffing up to become D5-branes. Accordingly, it should be given by an embedding $\rho^+ : \mathrm{SU}(2) \to \mathrm{SO}(2N)$. Indeed, if we decompose the real $2N$-dimensional representation of SO$(2N)$ in terms of irreducible representations of SU(2) as in (6.5), N_h for even h is even, because $\underline{h}$ for even h is pseudo-real. The global symmetry (6.26) is the commutant of this embedding ρ^+. Performing S-duality and exchanging D5-branes with NS5-branes, we obtain the quiver

$$\underline{\mathrm{SO}(2N)} - \mathrm{USp}(r_1) - \mathrm{O}(r_2) - \cdots - \mathrm{USp}(r_{J-1}) \tag{6.28}$$

where the underlined group is a flavor Higgs symmetry as before. Here J is always even, and the sizes are

$$r_a = \Big[\sum_{b=a+1}^{J} h_b \Big]_{+,-} , \qquad + : \mathrm{O}\ , \quad - : \mathrm{USp} \tag{6.29}$$

where $[n]_{+(-)}$ is the smallest (largest) even integer $\geq n$ $(\leq n)$. The two options refer

---

which is $\mathfrak{so}(k)$. At the level of the group, the projection selects the real subgroup of U(k), which is O(k).

to the group being O or USp. When the last group is USp(0), we remove it. These quivers have been introduced in [61] and called $T_{\rho^+}[\mathrm{SO}(2N)]$.

A negative puncture $\rho^- = \{h_1, \ldots, h_J\}$ before S-duality describes D3-branes on an O3$^+$ puffing up to become D5-branes. Accordingly, it should be given by an embedding $\rho^- : \mathrm{SU}(2) \to \mathrm{USp}(2N-2)$. Indeed, if we decompose the pseudo-real $(2N-2)$-dimensional representation of $\mathrm{USp}(2N-2)$ under SU(2), N_h for odd h is even, because $\underline{h}$ is strictly real when h is odd. The global symmetry (6.26) is the commutant of this embedding ρ^-. Performing S-duality, we get the 3d quiver

$$\underline{\mathrm{SO}(2N-1)} - \mathrm{USp}(r_1) - \mathrm{O}(r_2) - \cdots - \mathrm{O}(r_{\tilde{J}}) \qquad \text{with } \tilde{J} = [J]_+ \, . \tag{6.30}$$

The sizes are

$$r_a = \Big[1 + \sum_{b=a+1}^{J} h_b\Big]_{\tilde{\mp},-} \, , \qquad\qquad \tilde{\mp} : \mathrm{O} \, , \quad - : \mathrm{USp} \tag{6.31}$$

where $[n]_{\tilde{\mp}}$ is the smallest odd integer $\geq n$. The two options refer to the group being O or USp. These quivers are called $T_{\rho^-}[\mathrm{SO}(2N-1)]$.

They give rise to the S-dual pairs of boundary conditions

$$\begin{aligned} S:\ \big(\rho^+,\, 1,\, \varnothing\big)_{X,Y} &\ \mapsto\ \big(1,\, \mathrm{SO}(2N),\, T_{\rho^+}[\mathrm{SO}(2N)]\big)_{Y,X} \, , \\ S:\ \big(\rho^-,\, 1,\, \varnothing\big)_{X,Y} &\ \mapsto\ \big(1,\, \mathrm{O}(2N-1),\, T_{\rho^-}[\mathrm{SO}(2N-1)]\big)_{Y,X} \, . \end{aligned} \tag{6.32}$$

The Coulomb branch of $T_{\rho^+}[\mathrm{SO}(2N)]$ is $\mathcal{S}_{\rho^+} \cap \overline{\mathcal{N}} \subset \mathfrak{so}(2N)_{\mathbb{C}}$, whereas that of $T_{\rho^-}[\mathrm{SO}(2N-1)]$ is $\mathcal{S}_{\rho^-} \cap \overline{\mathcal{N}} \subset \mathfrak{usp}(2N-2)_{\mathbb{C}}$. As such, the symmetries on the Coulomb branch are given by the commutant of ρ^+ inside $\mathrm{SO}(2N)$ and of ρ^- inside $\mathrm{USp}(2N-2)$, respectively. They agree with the symmetries found from the brane construction, (6.26) and (6.27). The theories $T_\rho[\mathrm{SO}(r)]$ have a Higgs branch which is the closure of a certain nilpotent orbit $\rho^\vee$ of $\mathrm{O}(r)$.

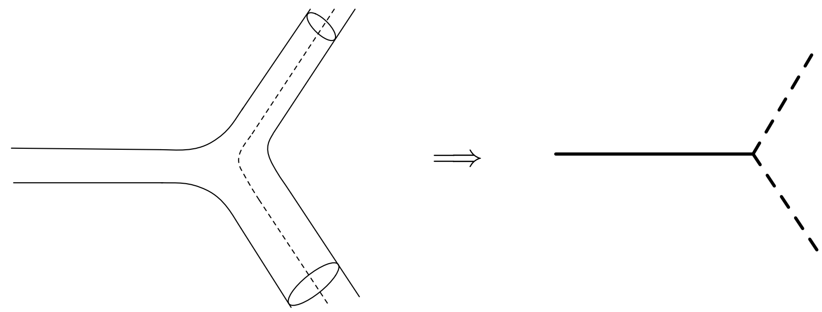


Fig. 35. 6d D_N theory on a pair of pants $\mathcal{C}$ with $\mathbb{Z}_2$ monodromy along two tubes. KK reduction along $S^1_{\mathcal{C}} \subset \mathcal{C}$ gives a junction of two segments of 5d USp$(2N-2)$ SYM and one of SO$(2N)$; further compactification on S^1_A gives its 4d version. Instead compactification on S^1_A gives 5d SO$(2N)$ SYM on a pair of pants with $\mathbb{Z}_2$ monodromy by the parity transformation $\sigma \in$ O$(2N)$; further KK reduction on $S^1_{\mathcal{C}}$ gives a junction of two segments of 4d O$(2N-1)$ SYM and one of SO$(2N)$. We obtain USp$(2N-2)$ in the first description and O$(2N-1)$ in the second, because the procedure involves the exchange of $S^1_{\mathcal{C}}$ and S^1_A, which acts as S-duality of 4d $\mathcal{N}=4$ SYM.

a. Two types of junctions and their S-duals

3d D_N Sicilian theories can be constructed by putting $\mathcal{N}=4$ SYM on a graph. We saw that there are two types of punctures: positive ones, which are boundary conditions for SO$(2N)$ SYM, and negative ones, which are boundary conditions for USp$(2N-2)$ SYM and create a twist line. Accordingly, to keep track of twist lines, on the segments of the graph we put either SO$(2N)$ or USp$(2N-2)$ SYM. We need to consider two types of junctions: a junction among three copies of SO$(2N)$ SYM, and a junction among one copy of SO$(2N)$ and two copies of USp$(2N-2)$. These junctions correspond to the maximal triskelions, see Figure 35. We will call the two resulting theories R_{2N} with SO$(2N)^3$ Higgs symmetry and $\widetilde{R}_{2N}$ with SO$(2N)\times$ USp$(2N-2)^2$ Higgs symmetry. When compactified on S^1, the boundary conditions at the junction are

$$\big(1,\, \mathrm{SO}(2N)^3,\, R_{2N}\big)_{X,Y} \qquad \text{and} \qquad \big(1,\, \mathrm{SO}(2N)\times\mathrm{USp}(2N-2)^2,\, \widetilde{R}_{2N}\big)_{X,Y}\ . \tag{6.33}$$

With these boundary conditions, all 3d D_N Sicilian theories can be reproduced via pants decomposition.

The S-dual of these boundary conditions can be easily obtained with the analysis in subsection c. We obtain

$$\begin{aligned}
S:\ &\big(1,\, \mathrm{SO}(2N)^3,\, R_{2N}\big)_{X,Y} &&\mapsto \big(1,\, \mathrm{SO}(2N)_{\mathrm{diag}},\, \varnothing\big)_{Y,X}\,, \\
S:\ &\big(1,\, \mathrm{SO}(2N)\times\mathrm{USp}(2N-2)^2,\, \widetilde{R}_{2N}\big)_{X,Y} &&\mapsto \big(1,\, \mathrm{O}(2N-1)_{\mathrm{diag}},\, \varnothing\big)_{Y,X}\ .
\end{aligned} \tag{6.34}$$

Here $\mathrm{SO}(2N)_{\mathrm{diag}}$ is the diagonal subgroup of $\mathrm{SO}(2N)^3$, while $\mathrm{O}(2N-1)_{\mathrm{diag}} \subset \mathrm{SO}(2N)\times \mathrm{O}(2N-1)^2$ corresponds to choosing an $\mathrm{O}(2N-1)$ subgroup of $\mathrm{SO}(2N)$, and then taking the diagonal subgroup of $\mathrm{O}(2N-1)^3$.

This can be proved also by considering a simple case in which we already know the mirror symmetry map. For instance, consider the 3d Sicilian theory given by one simple positive puncture on the torus: this is 3d $\mathcal{N}=8$ $\mathrm{SO}(2N)$ SYM. The graph construction has two $\mathrm{SO}(2N)$ segments, $(1, \mathrm{SO}(2N)^3, R_{2N})$ at the junction and $(\{2N-1,1\}, 1, \varnothing)$ at the puncture. The mirror theory is $\mathcal{N}=8$ $\mathrm{SO}(2N)$ SYM itself. The S-dual graph has $\mathrm{SO}(2N)$ on the segments and $(1, \mathrm{SO}(2N), \varnothing)$ at the puncture, because $T_{\{2N-1,1\}}[\mathrm{SO}(2N)]$ is an empty theory. To reproduce the mirror, we need $(1, \mathrm{SO}(2N)_{\mathrm{diag}}, \varnothing)$ at the junction. Similarly, consider the 3d Sicilian theory given by one simple positive puncture on the torus with a twist line around it: this is 3d $\mathcal{N}=8$ $\mathrm{USp}(2N-2)$ SYM. The graph construction has one $\mathrm{SO}(2N)$ and one closed $\mathrm{USp}(2N-2)$ segment, $(1, \mathrm{SO}(2N)\times\mathrm{USp}(2N-2)^2, \widetilde{R}_{2N})$ at the junction and $(\{2N-1,1\}, 1, \varnothing)$ at the puncture. The mirror theory is $\mathcal{N}=8$ $\mathrm{O}(2N-1)$ SYM. The S-dual graph has $\mathrm{SO}(2N)$ and $\mathrm{O}(2N-1)$ on the segments, and $(1, \mathrm{SO}(2N), \varnothing)$ at the puncture. To reproduce the mirror, we need $(1, \mathrm{O}(2N-1)_{\mathrm{diag}}, \varnothing)$ at the junction.

b. Mirror of Sicilian theories

Now it is easy to construct the mirrors of 3d Sicilian theories of type D_N obtained from an arbitrary punctured Riemann surface $\mathcal{C}$. First consider $\mathcal{C}$ of genus zero with only positive punctures. When we gauge two $T[\mathrm{SO}(2N)]$ together via their $\mathrm{SO}(2N)$ Coulomb symmetries, the Higgs branch of the combined theory is the cotangent bundle $T^*\mathrm{SO}(2N)_{\mathbb{C}}$, which has the action of $\mathrm{SO}(2N) \times \mathrm{SO}(2N)$ from the left and the right. This is broken to its diagonal subgroup on the zero-section. When it is gauged on both sides by different vector multiplets, the Higgs mechanism gets rid of one $\mathrm{SO}(2N)$ vector and the adjoint hypermultiplet. We are left with a star-shaped quiver with $\mathrm{SO}(2N)$ gauge group at the center. Then consider $\mathcal{C}$ with genus $g \geq 1$ and with only positive punctures. There will be g copies of $T^*\mathrm{SO}(2N)_{\mathbb{C}}$ gauged on both sides by the same $G = \mathrm{SO}(2N)$, *i.e.* G acts on the cotangent bundle by the adjoint action. Around the origin of $T^*\mathrm{SO}(2N)_{\mathbb{C}}$ all hypermultiplets are massless. We are left with a star-shaped quiver, with g extra $\mathrm{SO}(2N)$ adjoint hypermultiplets and $\mathrm{SO}(2N)$ gauge group at the center. The analysis so far was completely parallel to that of type A_{N-1} Sicilians.

Next, consider $\mathcal{C}$ of genus zero with n_+ positive and $2n_-$ negative punctures. When we gauge together two copies of $T[\mathrm{SO}(2N-1)]$ on the Coulomb branch, we get $T^*\mathrm{SO}(2N-1)_{\mathbb{C}}$ which spontaneously breaks the symmetry. However there will be $n_- - 1$ copies of $T^*\mathrm{SO}(2N)_{\mathbb{C}}$ which are gauged by two $\mathrm{O}(2N-1)$ on both sides: the gauge group is broken to the diagonal $\mathrm{O}(2N-1)$ and a hypermultiplet in the fundamental of $\mathrm{O}(2N-1)$ remains massless. We are left with a star-shaped quiver, with $n_- - 1$ extra fundamentals of the $\mathrm{O}(2N-1)$ gauge group at the center. See Figure 36a for the case $n_- = 2$.

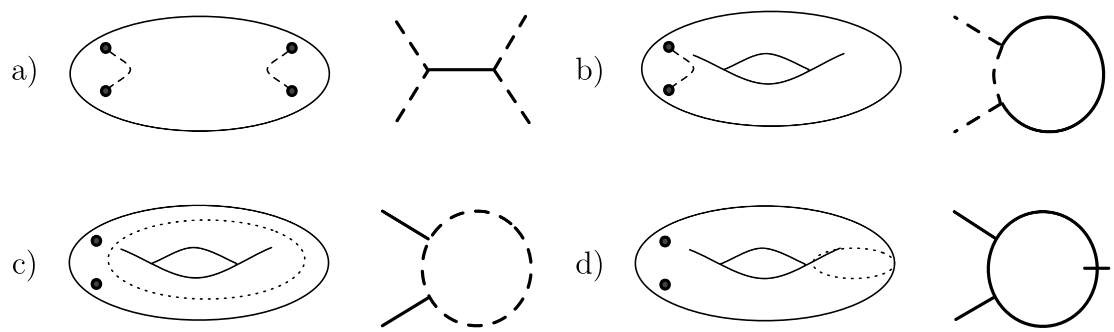


Fig. 36. a) The segment at the center gives $T^*\mathrm{SO}(2N)_{\mathbb{C}}$, to which $\mathrm{O}(2N-1)^2$ gauge groups couple. We are left with one $\mathrm{O}(2N-1)$ gauge group and one extra fundamental hypermultiplet. b) $T^*\mathrm{SO}(2N)_{\mathbb{C}}$ is coupled to $\mathrm{O}(2N-1)$ which acts by the adjoint action. We are left with an adjoint and a fundamental of $\mathrm{SO}(2N-1)$. c) With a monodromy when one crosses a big S^1, the resulting 4d SYM on a graph has segments with $\mathrm{O}(2N-1)$ gauge group. d) A monodromy when one crosses a small S^1 results in 4d SYM on a graph with a loop around which we have a monodromy. This is indicated as a mark in the graph shown on the right. c) and d) give rise to the same 3d theory in the low energy limit, as explained in the main text.

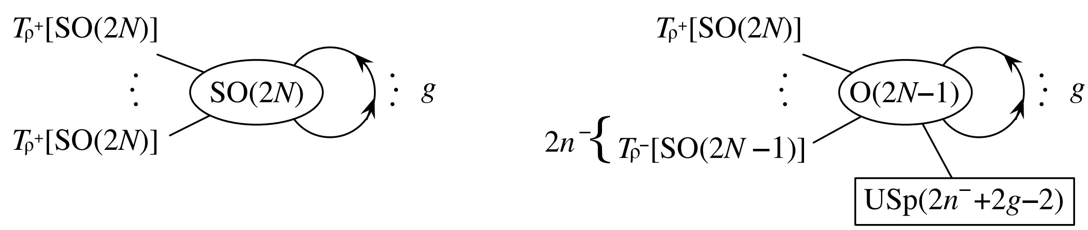


Fig. 37. Mirrors of D_N Sicilian theories of genus g. Left: Mirror of a D_N Sicilian with only positive punctures and no twist lines. The g adjoints carry USp($2g$) Higgs symmetry. Right: Mirror in the presence of some twist line, where $2n^- \geq 0$ is the number of negative punctures. The $n^- + g - 1$ extra fundamentals carry USp($2n^- + 2g - 2$) Higgs symmetry.

When $g > 0$, there are many choices for the configuration of monodromies: they are classified by $H_1(\mathcal{C} \setminus \{\text{punctures}\}, \mathbb{Z}_2)$. When one has two negative punctures but without twist lines on a handle (see Figure 36b), $T^*\mathrm{SO}(2N)_{\mathbb{C}}$ is gauged by the same $\mathrm{O}(2N-1)$ via the adjoint action and we get one adjoint and one fundamental of $\mathrm{SO}(2N-1)$. Another possibility is to have a closed twist line along a handle of the graph (Figure 36c): $T^*\mathrm{SO}(2N-1)_{\mathbb{C}}$ is gauged on both sides by the same $G = \mathrm{O}(2N-1)$ via the adjoint action, giving rise to an adjoint of $\mathrm{SO}(2N-1)$. If we take the S-duality of the 4d Sicilian theory first and then compactify it down to 3d, we get the 4d SYM on a graph shown in Figure 36d. This amounts to gauging $T^*\mathrm{SO}(2N)_{\mathbb{C}}$ with one $\mathrm{SO}(2N)$ with the embedding

$$\mathrm{SO}(2N) \ni g \mapsto (g, \sigma g \sigma) \in \mathrm{SO}(2N) \times \mathrm{SO}(2N) \tag{6.35}$$

where $\sigma \in \mathrm{O}(2N)$ is the parity transformation. The theory spontaneously breaks the gauge group to $\mathrm{O}(2N-1)$, which is the subgroup of $\mathrm{SO}(2N)$ invariant under parity, and eats up $2N-1$ hypermultiplets. We are left with $\mathrm{O}(2N-1)$ with just one adjoint.

Summarizing, consider a 3d Sicilian theory defined by a genus g Riemann surface $\mathcal{C}$, some number of punctures (of which $2n_-$ are negative) and possibly extra closed twist lines in $H^1(\mathcal{C}, \mathbb{Z}_2)$. If there are no twist lines at all (so $n_- = 0$), the mirror is a star-shaped quiver where an $\mathrm{SO}(2N)$ group gauges together all positive punctures and g extra adjoint hypermultiplets. If there are twist lines, the mirror is a star-shaped quiver where an $\mathrm{O}(2N-1)$ group gauges together all punctures, g extra adjoints and $(n_- + g - 1)$ extra fundamentals. This is summarized in Figure 37. It is reassuring to find that the resulting mirror theory does not depend on the pants decomposition.

B. More 3d mirror pairs

1. Review

Three dimensional $\mathcal{N} = 4$ gauge theory has $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R symmetry. This can be seen from the compactification of 6d $\mathcal{N} = 1$ theory: $\mathrm{SU}(2)_R$ is the R symmetry of the 6d theory while $\mathrm{SU}(2)_L$ symmetry comes from the rotation group of the three dimensional space on which we do the reduction. The moduli space of the vacua has Coulomb branch and Higgs branch (we also have the mixed branch). The Higgs branch is a Hyperkahler manifold whose Kahler form transforms under $\mathrm{SU}(2)_R$ and invariant under the $\mathrm{SU}(2)_L$. There usually are global symmetries acting on Higgs branch, when we have a lagrangian description, the global symmetry can be read readily, we can turn on mass terms and preserve $\mathcal{N} = 4$ supersymmetry. The Coulomb branch is also a Hyperkahler manifold whose kahler form transforms under $\mathrm{SU}(2)_L$ and invariant under $\mathrm{SU}(2)_R$. Usually there is only a $\mathrm{U}(1)$ global symmetry arising from the shift symmetry of the photon, but sometimes the symmetry is enhanced due to monopole operators [61, 70]; if there are $\mathrm{U}(1)$ factors in gauge group, we can turn on Fayet-lliopoulos (FI) terms and preserve the same number of supersymmetry. For some theories, the Higgs branch and Coulomb branch intersects at a single point, and there is an interacting SCFT on which both $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ acts. This SCFT is the IR fixed point under the RG flow of the theory.

Suppose we have two three dimensional $\mathcal{N} = 4$ theory A and B, and both theories flow to non-trivial IR fixed points $\mathcal{A}$ and $\mathcal{B}$. We say they are mirror pairs if the Higgs branch of $\mathcal{A}$ is identical to the Coulomb branch of $\mathcal{B}$ and vice versa [60]. The mass terms are identified with the FI terms under mirror symmetry. Since Coulomb branch gets quantum corrections and Higgs branch has the non-renormalization property and is exact by doing classical calculation, the quantum effects of one theory is captured by classical effects of another theory. The IR fixed points are usually strongly coupled, we mainly use the UV theory A and B to learn their IR behavior and simply states the theory A and B are mirror pairs.

In [25], a large class of mirror pairs are found. Theory A arises from compactifying 4d $\mathcal{N}=2$ generalized superconformal quiver gauge theory on a circle; Theory B is a star-shaped quiver. Four dimensional theory is realized as compactifying six dimensional $(0,2)$ theory on a Riemann surface Σ with punctures which are classified by Young Tableaux [11]. The Hitchin's equation defined on Riemann surface is the BPS equation and whose moduli space with specified boundary condition at the puncture is the Coulomb branch of the three dimensional theory. The boundary condition of the Hitchin's equation is a regular singularity for this class of theories. The Hitchin's moduli space can be approximately by a quiver as discovered by Boalch [58], it turns out that this quiver is the mirror quiver for the theory A. We only consider those theories for which the Hitchin's system is irreducible. In physics language, this means that the quiver gauge theory has a dimension N operators in the Coulomb branch.

In general, we can not write a lagrangian description for the theory; The weakly coupled gauge group and flavor symmetries can be determined using the information on the puncture , we also know the flavor symmetry. These theories and its generalization are further studied in [66, 71, 44, 72, 73, 74]. Various S-duality frames of 4d theory are identified with the different degeneration limits of the punctured Riemann surface.

We further compactify theory A on a circle S and get a 3d $\mathcal{N}=4$ theory. The compact space is $\Sigma \times S$. We can model each leg in pants decomposition of the punctured Riemann surface as a cylinder $S^1 \times I$, then the three dimensional space we do the reduction on this leg is $(S^1 \times I) \times S$. We can change the order of compactification and regard the three space as $(S^1 \times S) \times I$: in first step, we get a

4d $\mathcal{N} = 4$ SU(N) SYM and we assume that the boundary condition at the ends of I is classified by the same Young Tableaux. This fact can be seen from the following argument: the Hitchin's equation around the singularity is identified with the Nahm's equation with specified singular boundary condition, which is exactly the equation governing the boundary condition for $\mathcal{N} = 4$ SU(N) SYM on the half space. In this order of compactification, 3d theories are represented as 4d $\mathcal{N} = 4$ SYM on a one dimensional graph. In fact, the graph is just the dual graph of the punctured Riemann surface as described in [23], it is a trivalent graph with lots of three junctions.

With this graph representation of the theory A, the mirror symmetry is understood as the S duality of the $\mathcal{N} = 4$ on the graph. The S duality of $\mathcal{N} = 4$ SYM on half space has been studied in full detail in [68, 61], in particular, the dual of the boundary condition we discussed earlier is worked out. The mirror of each boundary condition is a quiver leg. For example, if the Young Tableaux has heights $[h_1, h_2, ...h_r]$ with $h_1 \geq h_2 \geq ... \geq h_r$, then the mirror quiver leg is

$$N - U(n_1) - U(n_2) - .... - U(n_{r-1}) \tag{6.36}$$

where $n_i = \sum_{i+1}^{r} h_j$, and the first N means we have a global SU(N) flavor symmetry.

The S-dual of the three junctions is worked out in [25, 44], it is simply the diagonal part of three SU(N) gauge groups on the legs connecting with the junction. By combining various components, the mirror theory is just a star-shaped quiver with a SU(N) node at the center. In another word, we simply gauge together the SU(N) node of each leg. It is interesting to note that the mirror does not depend on the pants decomposition.

Let's give an example to illustrate the main idea. Consider four dimensional $\mathcal{N} = 2$ SU(2) gauge theory with four fundamentals. It is derived from six dimensional theory on a Riemann sphere with four punctures. One of the pants decomposition is

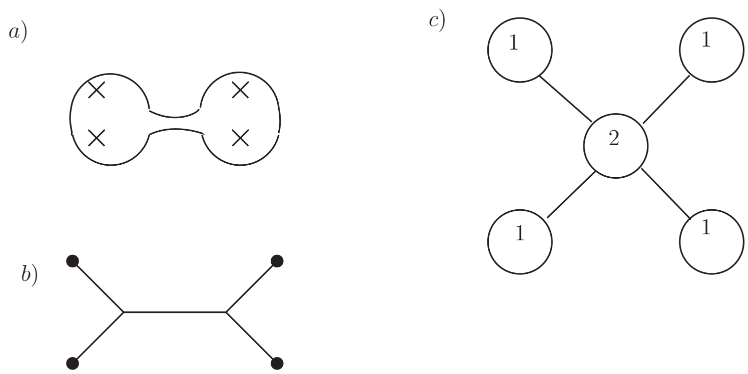


Fig. 38. a): One S-duality frame for four dimensional $\mathcal{N} = 2$ SU(2) with four fundamentals, each puncture carries a SU(2) flavor symmetry. b): Three dimensional version of (a), which is derived by compactifying (a) on a circle. we represent it as $N = 4$ SYM on the graph. c): The graph mirror of (b), which is simply derived by gluing the SU(2) flavor symmetry of four quiver tails.

described in Figure 38a). The graph representation of 3d theory is shown in Figure 38b). The internal leg represents the SU(2) gauge group. The Young Tableaux of the boundary condition has the heights $[1, 1]$, the mirror of this boundary condition is a quiver tail $2 - \mathrm{U}(1)$, after gauging the common SU(2) node, we found the mirror in Figure 38c).

We can extend the above analysis to D_N theory [25]. Since the four dimensional gauge theory involves not only SO group but also USp group, we need to turn on the z_2 monodromy line. In the pants decomposition, if there is a monodromy around the circle S^1, then the gauge group on that leg would be $\mathrm{SO}(2N + 1)$. To get a three dimensional theory, we further compactify the theory on S. To represent the theory as $N = 4$ SYM on the graph, we change the order of compactification, we first compactify six dimensional theory on $S \times S^1$, the four dimensional gauge group is $\mathrm{USp}(2N - 2)$ which is the S dual of the theory derived from $S^1 \times S$, the boundary condition at the ends should be given by the Young Tableaux of USp($2N-2$) which is exactly the case as found in [44]. There are two types of junctions: the first type is the one for which there is no USp leg while the second type has two USp legs. The S-dual of the boundary condition of SO group and USp group has been also studied in [61]. The S-dual of two different junctions are worked out in [25]: the dual is the diagonal SO($2N$) group for the first type of junction while the S-dual of the second type is a diagonal USp($2N-2$) group. The dual of the boundary conditions are also worked out explicitly in [61].

2. Adding more fundamentals

a. A_{N-1} theory

Genus 0 theory The theories studied in [25] is superconformal in the four dimensional sense. It is interesting to extend to the non-conformal cases, i.e. those theories with more fundamentals on the weakly coupled gauge group(we call them theory $\tilde{A}$). We will use the graph representation of the three dimensional theory we reviewed in last subsection. Before doing that, we want to introduce some important concepts on 3d quiver gauge theories.

Since the IR theory we want to study is usually strongly coupled, we hope we can learn some of its property from the UV theory, this is not always possible, for instance, there might be accidental R symmetry in the IR which is not the same R symmetry in the UV. Consider 3d $\mathcal{N}=4$ $SU(N_c)$ theory with N_f fundamentals, let's define the excess number of it:

$$e = N_f - 2N_c. \tag{6.37}$$

This theory is called "good" if $e \geq 0$, "ugly" if $e=-1$, "bad" if $e<-1$. For the "good" theory, there is a standard critical points and the IR R symmetry is just the

R symmetry in the UV theory. For the “ugly” theory, the IR limit is just a set of free hypermultiplets. For the “bad” theory, the IR limit is not a standard critical point and the R symmetry might be accidental symmetry. For the ”good” theory, the theory can be completely higgsed and there is a pure Higgs branch, we can learn a lot about the IR limit from the UV theory. We mainly focused on the ”good” theory in this subsection. If $e = 0$, we call it a “balanced” theory which has interesting property on the Coulomb branch.

The above definition can be extended to a quiver. We call a quiver “good” if $e_i \geq 0$ for every node. The Coulomb branch symmetry is enhanced due to the monopole operators. If we have a linear chain of balanced quiver with P nodes, i.e. $e_i = 0$ for every node, then the global symmetry on Coulomb branch is enhanced to $SU(P+1)$. If the balanced quiver has the shape D_n or E_n type dynkin diagram, then the symmetry is enhanced to the corresponding D_n or E_n group. The global symmetry for a general “good” quiver is just the product of enhanced non-abelian symmetries and abelian U(1)s from non-balanced nodes. This is useful since we can read the exact global symmetry of Higgs branch of the theory A using the mirror. For instance, for the theory SU(2) with four fundamentals, the flavor symmetry is SO(8). In the Gaiotto’s representation in Figure 38a), only $\mathrm{SU}(2)^4$ subgroup is manifest, while we can see the full SO(8) symmetry in the Coulomb branch of the mirror using monopole operators. This example might be trivial since we have a lagrangian description for A, but for other strongly coupled theory, the mirror theory is very useful to see full flavor symmetries.

For the irreducible theory A we considered in this paper, the mirror B is always good as one can check. Now let’s consider theory $\tilde{A}$ which is derived by adding more fundamentals to the gauge groups of the theory A considered in [25], the mirror $\tilde{B}$ should also be a good quiver. There is a graph representation for A as we reviewed

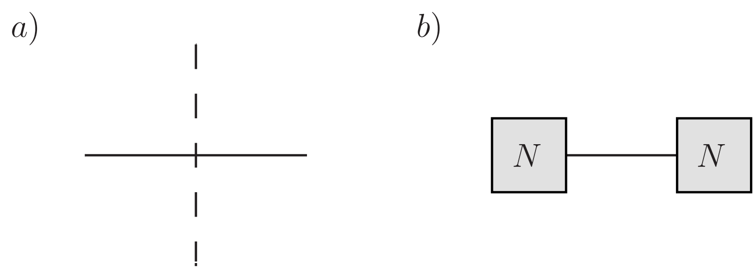


Fig. 39. Left: The addition of a "D5" brane to the internal leg of the graph. Right: Its mirror.

in last subsection, the gauge groups are represented as the internal legs. To add fundamentals, it is natural to think adding some "D5" branes on the internal leg. There is one important question: Is the IR limit of $\mathcal{N} = 4$ SYM on the graph the same as the IR limit of theory $\tilde{A}$? In general the answer is not: there are some free hypermultiplets in the IR besides the fix point theory of $\tilde{A}$. This can be seen from the mirror of the graph. The graph mirror is in general a "bad" quiver.

The mirror of the graph is simple, the S-dual of the "D5" branes are "NS" brane. From the gauge theory point of view, there are now two U(N) gauge groups connected by a bi-fundamental, see Figure 39. In general, the graph mirror is "bad" which reflects the fact the IR limit of the graph does not coincides with the IR limit of theory $\tilde{A}$. We want to extract the mirror $\tilde{B}$ from the graph mirror.

The process we suggest is the following: for any "bad" or "ugly" node on the graph mirror with the excess number $e_i < 0$, we replace its rank by

$$n_i^{'} = n_i + e_i, \tag{6.38}$$

then the excess number of the quiver nodes around it will also be changed, if there are still some "bad" nodes, we will do the same manipulation on those nodes. We continue doing this until all the nodes are "good", the resulting theory is the mirror $\tilde{B}$ for the theory $\tilde{A}$.

The theory $\tilde{B}$ should have the same Higgs branch as the star-shaped quiver B. Since theory $\tilde{A}$ has the same Coulomb branch dimension as A. The graph mirror has the same Higgs branch as B as we can easily count: we add one bi-fundamental and one U(N) node, the net contribution to Higgs branch is zero. The graph mirror has two central nodes, and the only possible "bad" node is one of the central node, See Figure 40. for the illustration. The contribution of this central node to the Higgs branch is

$$N_f N_c - N_c^2 = N_c(N_f - N_c). \tag{6.39}$$

This number is unchanged if we change the rank of the gauge group to $N_c^{'} = N_f - N_c$ and keep N_f unchanged. The number $N_c^{'} = N_c + N_f - 2N_c = N_c + e$ which is just the number we defined earlier. $N_c^{'}$ should be less or equal than N, so we can only do the manipulation for those quiver node with $e < 0$.

We want to point out some generic features of the manipulation. As we noticed earlier, only one of the central node can be "bad" for the graph mirror, the excess number of it is $e < 0$. After changing its rank, its new excess number is $-e$. The excess number of its adjacent nodes are increased by e. If none of those new excess numbers are negative, then our manipulation stops, there is one more U(1) global symmetry on the Coulomb branch from the new node. This reflects the fact there is an extra U(1) flavor symmetry coming from the new added fundamental. It is possible some of the adjacent node becomes "balanced" and therefore we have enhanced symmetry, however we can only have one new "balanced" adjacent node with just one exception.

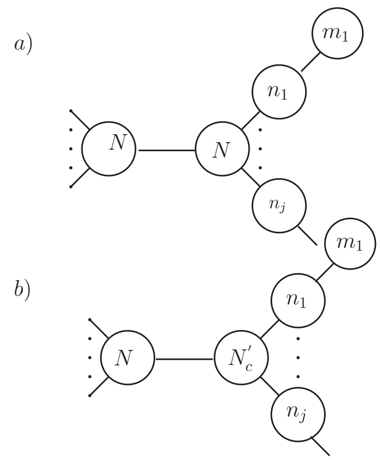


Fig. 40. Top: The naive mirror for adding one fundamental to one of weakly coupled gauge group; We assume that the left central node is bad. Bottom: We replace the rank of the left central node with $N_c^{'} = N_f - N = \sum_{k=1}^{j} n_k$.

We order the rank of the adjacent nodes so that $n_1 \leq n_2 \leq ... \leq n_j$, so

$$e_i^{'} \geq \sum_{k=1}^{j} n_k - 2n_i, \tag{6.40}$$

$e_i^{'} \geq 0$ for any $i \geq 3$. $e_2 \geq 0$, it is zero only when there are only two identical quiver tails with Young Tableaux $[N-n_1, n_1]$. This is consistent since after adding only one fundamental, the flavor symmetry can only be changed from SU(k) to SU($k+1$) or from SO($k+2$) to SO($2k+2$) (this is for the USp group).

It is possible that n_1 node becomes "bad" after we change the rank of the central node. We will focus on this quiver tail. The excess number of each node on this quiver tail can be read from the Young Tableaux:

$$e_i = n_{i+1} + n_{i-1} - 2n_i = \sum_{j=i+2}^{r} h_j + \sum_{j=i}^{r} h_j - 2\sum_{j=i+1}^{r} h_j = h_i - h_{i+1}, \tag{6.41}$$

we take $h_{r+1} = 0$ (here we use i to denote the node on this particular quiver tail and

n_i as its rank), the excess number is non-negative as from the definition of the Young Tableaux. It is a several chains of balanced quiver separated by the "good" quiver nodes.

After changing the rank of this node, the new excess number of the central number is $e_c^{'} = e_1$ which is positive. The excess number of the n_1 is $e_1^{'} = -e_1 - e$. The new excess number of other adjacent node n_2 is $e_2^{'} = e_1 + e_2 + e$, if this number is non-negative, then our process stops. If not, we continue the process, the excess number of the first node changed to e_2 though. We only need to do one manipulation on each possible node. The general conclusion is that the process stops at the jth node on the quiver tail with the condition

$$e_1 + e_2 + ...e_j + e \geq 0, \quad e_1 + e_2 + ..e_{j-1} + e < 0. \tag{6.42}$$

In particular, the structure of the balanced chain is not changed. The final form of the quiver with its rank and excess number is shown in Figure 41. No new rank number is zero or negative, since

$$n_i^{'} = n_i + e + e_1 + ...e_i = n_i + h_1 - h_{i+1} + e > n_i + h_1 - h_{i+1} + n_1 - N = n_i - h_{i+1} > 0. \tag{6.43}$$

This ensures that on quiver node disappears. The structure of the new quiver shows that there is a extra U(1) on the symmetry of the Coulomb branch, which is exactly what we want.

For four dimensional theory, the gauge group contents depend on the pants decomposition of the Riemann surface; To add fundamentals to the gauge group, we must specific the pants decomposition. Go to three dimensions, the mirror is obviously different for different pant decomposition, which is in contrast with conformal case for which the mirror is independent of pants decomposition. This allows us to determine different duality frames of 4d SCFT as we will see later.

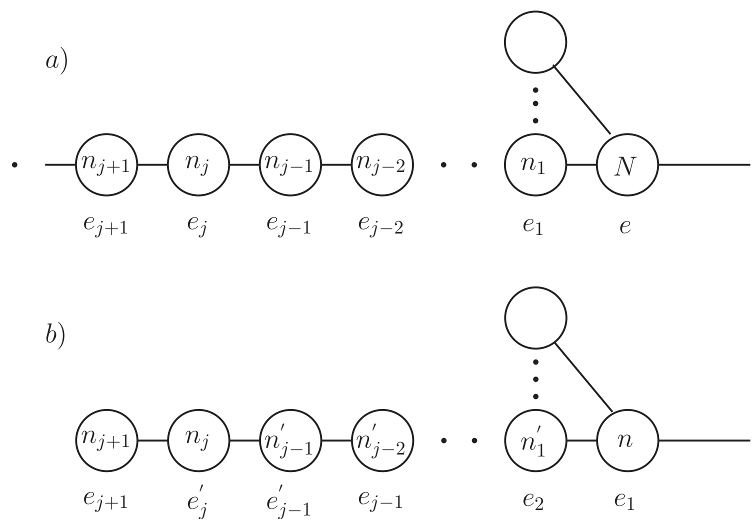


Fig. 41. Top: The rank and excess number of a quiver tail associated with a central node, we assume that $e + e_1 < 0$ for this quiver tail. Bottom: The rank and excess number of the quiver tail after the manipulation is finished. It stops at the jth node, from the condition, we have $e_j > 0$, this shows that no balanced node is lost, and they balanced chain is not altered. The new excess number is $e'_{j-1} = -(e + e_1 + e_2 + ...e_{j-1}) > 0$, $e'_j = e + e_1 + e_2 + ...e_j \geq 0$, so it is only possible for one more balanced node to appear. If a new balanced node appears, it shows that there are already fundamentals exist; The new rank is $n'_i = n_i + e + e_1 + ...e_i$.

Let's give an example to illustrate our main idea. The four dimensional theory is the original example studied by Argyres-Seiberg [10], in one weakly coupled duality frame A_1, it is just a SU(3) theory with six fundamentals; In another duality frame A_2, there is a weakly coupled SU(2) gauge group coupled with one fundamental and E_6 strongly coupled theory. The S-duality can be understood from the six dimensional construction. See Figure 42 for the pants decomposition and the graph representation for the corresponding three dimensional theory.

Now let's add more fundamentals (say two as in Figure 43) to the gauge groups of the above theories, namely, we now consider theories $\tilde{A}_1$ and $\tilde{A}_2$. To find their mirrors, we use the graph representation of the conformal theories and add "D5" branes on the internal leg, we apply the S-dual and find graph mirror. If the mirror quiver is "good", then this quiver is just the mirror of $\tilde{A}$; if the mirror is "bad", this means that the IR limit of the graph is not the same as the theory $\tilde{A}$, but we can do the manipulation as we described earlier to find the mirror of $\tilde{A}$.

The graph representation and graph mirrors are shown in Figure 43. The simple puncture is represented by the Young Tableaux with heights $[2,1]$, the quiver tail to it is just $3-\mathrm{U}(1)$; The circle cross has partition $[1,1,1]$, the quiver tail is $3-\mathrm{U}(2)-\mathrm{U}(1)$. The mirror of "D5" branes is to cut the gauge groups into two and introduce a bi-fundamental connecting them.

The quiver in Figure 43(a) is "good", so we conclude that the graph representation on the left of Figure 43(a) has the same IR limit as the three dimensional SU(3) with 8 fundamentals, and the graph mirror is just the mirror of theory $\tilde{A}_1$, this is in agreement with the result in [62].

The quiver in Figure 43b) is "bad": the SU(3) node on the left has excess number negative 1, so we replace it with a $U(2)$ node, then the SU(3) node adjacent to it becomes "ugly" with excess number negative 1, we also replace it with $U(2)$. After

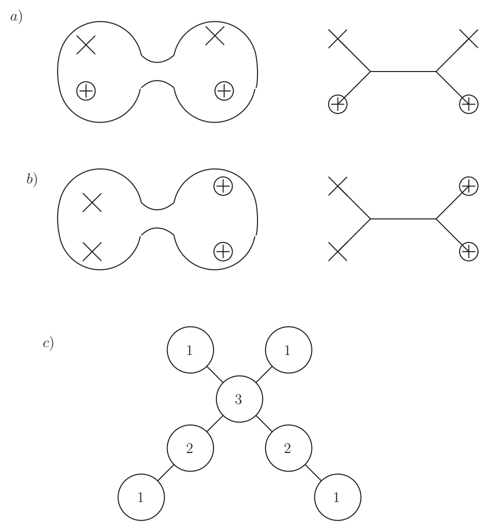


Fig. 42. a): The weakly coupled duality frame with SU(3) gauge group on the left, there are two type of punctures: the cross represents the simple puncture with Young Tableaux $[2, 1]$, the circle cross represents the full puncture with Tableaux $[1, 1, 1]$. The graph representation for three dimensional theory is shown on the right. b): The weakly coupled duality frame with SU(2) gauge group on the left, graph representation for three dimensional theory on the right. c): The mirror for theory (a) and (b), they are identical.

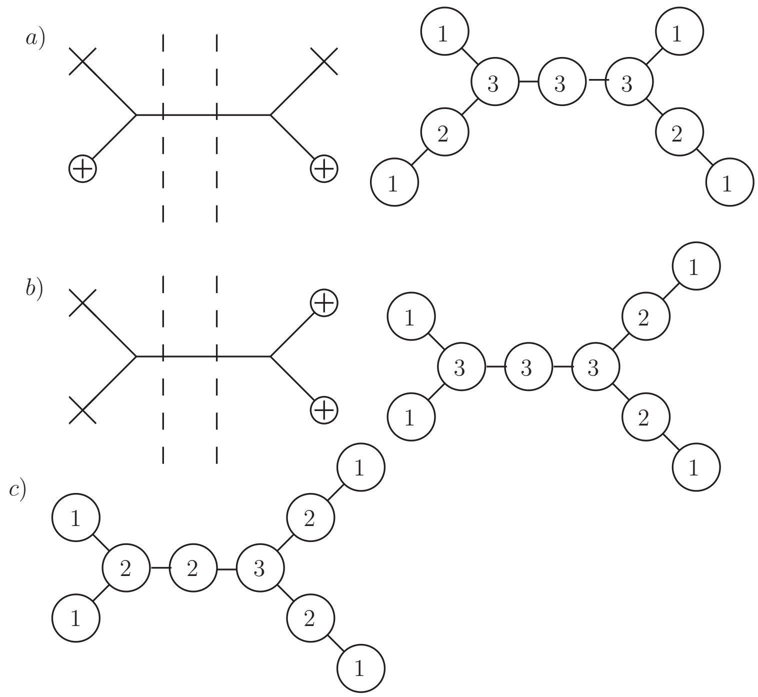


Fig. 43. a): On the left, we add two "D5" branes on the internal segment which represents a SU(3) gauge group, on the right, we apply S-duality and get a good quiver. b): We add two "D5" branes on the SU(2) gauge group, the left-hand side is the mirror quiver which is bad, the left central $U(3)$ node is "bad". c): We replace the bad $U(3)$ node with U(2), and then replace the adjacent U(3) with U(2) node, the resulting quiver is "good".

doing this, we get a "good" quiver as shown in Figure 43c), this is the mirror for the theory $\tilde{A}_2$.

Let's do some check on our result. The theory $\tilde{A}_2$ has the same Coulomb branch dimension (we always mean the hyperkahler dimension in this paper) as A_2 and the Higgs branch dimension of $\tilde{A}_2$ is increased by four. Comparing the quiver in Figure 43c) with the quiver in Figure 42c), its Coulomb branch dimension is increased by 4 and Higgs branch dimension is not changed. The flavor symmetry of $\tilde{A}_2$ is SO(6) × SU(6). The SO(6) is from three fundamentals while SU(6) is from E_6 matter.

In the Figure 43c), on the left, we have a linear chain of three balanced quiver and on the right we have a linear chain with 5 balanced quiver, so the symmetry on the Coulomb branch is SU(4) $\times$ SU(6) which is the same as the symmetry on the Higgs branch of $\tilde{A}_2$. (Notice the U(1) symmetry on the middle U(2) node is decoupled).

In fact, we can use the "D5" brane as a probe to find out what is the weakly coupled gauge group SU(k) (or $USp(k)$ in some cases) for 4d SCFT by counting the change of the Coulomb branch dimension of the mirror. Since for the theory $\tilde{A}$, the Higgs branch is increased by k, if we know the change of Coulomb branch dimension of the mirror, we can determine the weakly coupled gauge group. To determine whether it is a USp group of SU group, we can see the enhanced symmetry on the Coulomb branch of the mirror. If the mirror quiver has a balanced part with shape of D_n dynkin diagram, then the gauge group is USp(k), otherwise the weakly coupled gauge group is SU(N). The weakly coupled gauge group can also be determined using the degeneration limit in [23]. Here we use three dimensional mirror symmetry to do the job. We describe one example in Figure 44 Comparing the quiver in Figure 44c) and Figure 44a), the Coulomb branch dimension is increased by 4, since the quiver does not have a balanced part with D_n dynkin diagram shape, the gauge group is SU(4), this is in agreement with the result using the degeneration method as described in [23]. A special case is if the graph mirror is a "good" quiver, then the gauge group is SU(N) or USp(N) as the Coulomb branch of the mirror is increased by N.

We can also find out how many fundamentals on the gauge group for the four dimensional conformal theory. Since if there are l fundamentals exists, after adding one more fundamental, the global symmetry on Higgs branch is enhanced from U(l) to U($l+1$). In the mirror, we can see the change of the global symmetry on Coulomb branch using monopole operators, and we can determine k. In the quiver of Figure 44a), the global symmetry on Coulomb branch is SU(6)$\times$U(1)$\times$SU(2); For the quiver

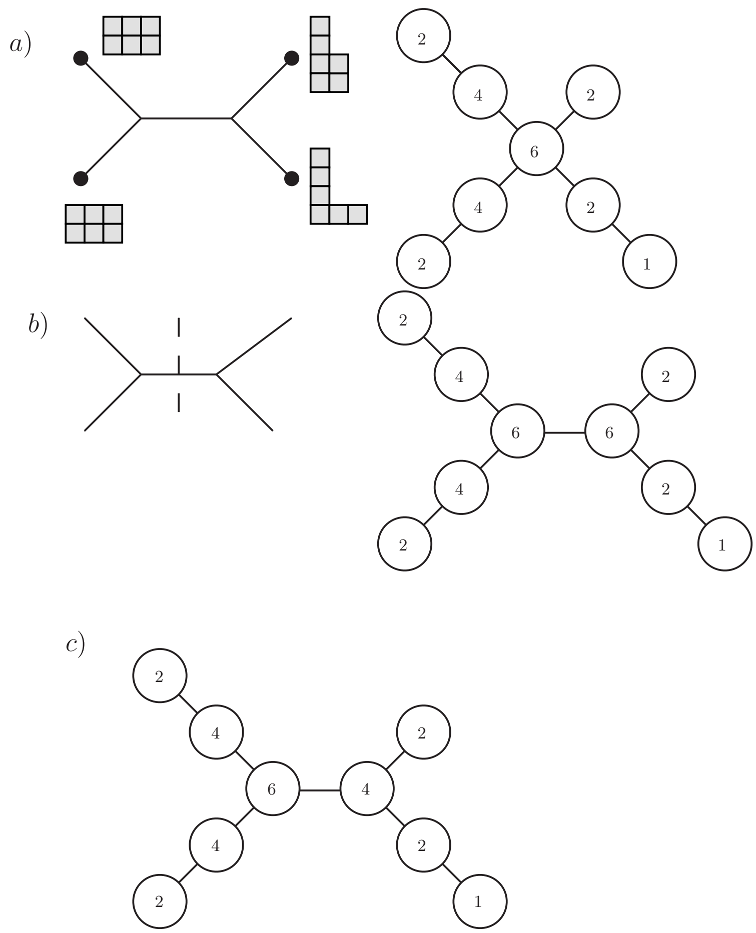


Fig. 44. a): A generalized quiver gauge theory A in three dimensions, its graph representation is depicted on the left, we draw Young Tableaux for the boundary condition, its mirror is depicted on the right. b): We add one "D5" branes on internal leg of the graph in (a), the mirror of the graph is shown on the right. c): The mirror of the theory $\tilde{A}$ which has one more fundamental than A.

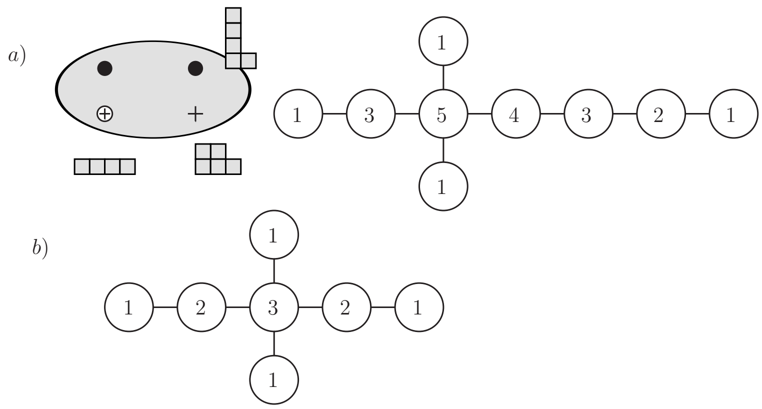


Fig. 45. Top: A four dimensional theory is derived by compactifying six dimensional A_4 theory on a sphere with four punctures, the Young Tableaux of the punctures are depicted, its 3d mirror can be found from the information in the puncture, it is a bad quiver, the central node has negative excess number. Bottom: For the bad node, we change its rank using the formula $N_c' = N_c + e$, after this modification, the excess number of the quiver nodes around the central are also changed and they are "bad", we do the modification on those nodes too, at the end, we get a "good" quiver, this quiver is the same as the quiver depicted in Figure 4.

in Figure 44c), the symmetry on Coulomb branch is SU(6) × SU(2) × U(1) × SU(2), we see the global symmetry is changed from U(1) to U(2), so originally we have only 1 fundamental. This is also in agreement with the result in [23].

There is another application of our procedure of extracting irreducible theory from "bad" quiver. We consider four dimensional irreducible theory up to now. For the reducible theory, the three dimensional mirror is "bad", which means that there are free hypermultiplets besides the SCFT, those SCFT are actually represented irreducibly by lower rank six dimensional $(0, 2)$ theory. Using our method, we can extract the free hypermultiplets and the irreducible SCFT from the 3d mirror. Those reducible theories have been considered in [74], our method gives another simple way to study them.

For instance, consider A_4 theory on a sphere with four punctures which are labeled as $[4,1],[4,1],[1,1,1,1],[2,2,1]$ (This is the first example in the appendix of [74]). The 3d mirror of this theory is depicted in Figure 45a), it is a bad quiver, so we do the modification to the rank of the bad quiver node, finally, we get a good quiver which is depicted in Figure 45b). One can recognize that this good quiver is the same as the quiver in Figure 42, so we can conclude that this is the SCFT part of the theory. Comparing the quiver in Figure 45a) and Figure 45b), the Coulomb branch is decreased by 10. So we conclude that the theory composes of a SCFT as described in Figure 45 and 10 free hypermultiplets. This is in agreement with the result in [74].

Let's go back to four dimensional $\mathcal{N}=2$ generalized quiver gauge theory. We have shown how to determine the weakly coupled gauge group and the number of fundamentals on it by using the "D5" brane probe. However, there are other strongly coupled matter systems coupled with the gauge group, we want to determine them. Let's follow the procedure in [23]: the weakly coupled gauge group corresponds to the long tube of the Riemann surface; we first consider the gauge group at the end of the quiver and completely decouple this gauge group, two new punctures appear, the Riemann surface are decomposed into two parts: a three punctured sphere Σ_1 and another sphere Σ_2 with a lot of punctures, see Figure 46a). The information of the matter system can be read from the three punctured sphere, we also want to determine what is the new puncture p on Σ_2. In [23], we assume that the two new appearing punctures are identical and find the new puncture by counting the Coulomb branch dimension. Motivated by our study of mirror symmetry in this paper, we follow a different approach, we assume the new puncture on Σ_1 is always the maximal puncture.

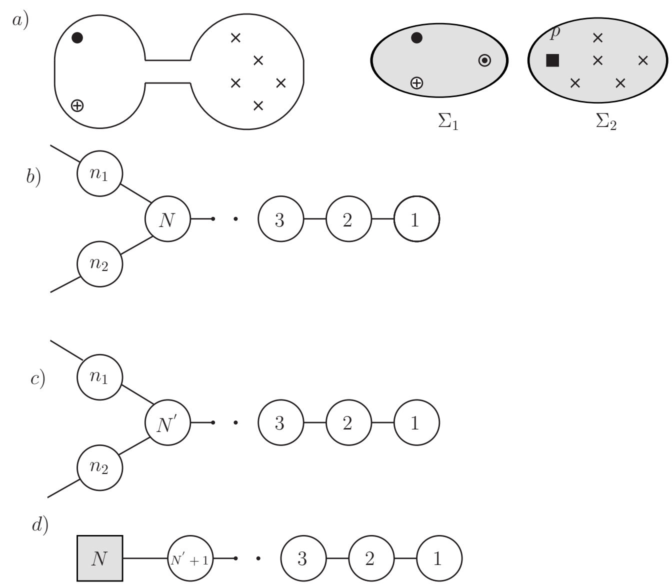


Fig. 46. a): A four dimensional $\mathcal{N}=2$ generalized quiver gauge theory is derived from six dimension $(0,2)$ SU(N) SCFT on this punctured sphere, the gauge group is represented by a long tube; We show one weakly coupled gauge group at the end of the quiver; After completely decoupling the gauge group, the Riemann surface becomes into two parts Σ_1 and Σ_2, we put a full puncture at the new formed three punctured sphere. There is a new puncture p on Σ_2. b): The 3d mirror for three punctured sphere. c): If the mirror in (b) is "bad", we change the rank of the "bad" nodes and get a "good" quiver, we assume that n_1 and n_2 nodes are "good". d): The mirror quiver tail for the puncture p for the case (c).

We extract the matter information by looking at the 3d mirror of the three punctured sphere B, see Figure 46b) , If the mirror is "good", then the matter system is just a strongly coupled isolated SCFT and the puncture p is just the maximal puncture. The weakly coupled gauge group is just SU(N) as we have shown earlier. If the mirror quiver is "bad", we can still extract the matter information by doing the manipulation we have used to find the mirror for the theory with more fundamentals. There are several situations we need to consider. The first situation is that after changing the rank of the central node to n_1+n_2-1, the n_1 and n_2 nodes are "balanced" or "good", in this case, the resulting quiver M_3 is shown in Figure 46c). This is the matter coupled to the gauge group. The quiver tail for the new puncture p is shown in Figure 46d). There are some checks on our result. The weakly coupled gauge group is $SU(n_1+n_2)$ from our previous analysis; For the quiver Figure 46d), there is indeed a chain of n_1+n_2-1 "balanced" nodes which has an enhanced $SU(n_1+n_2)$ symmetry on Coulomb branch. The original quiver is formed by gauging this global symmetry and the $SU(n_1+n_2)$ symmetry of the quiver p. Another serious check is to compare the Coulomb branch and Higgs branch dimension of the decomposed system and generalized quiver, they are in agreement with each other (the calculation is the same as we have done in [23], though a little bit tedious).

The other cases are more complicated. One usually have both free fundamental hypermultiplets and strongly coupled matter system. We already know how to see fundamentals, using the above method one can also extract the strongly coupled matter system.

Higher genus theory Let's next consider the theory associated with the higher genus Riemann surface. The theory A is a generalized quiver gauge theory and the mirror theory B has adjoint matter attached to the central SU(N) node. There is no complete Higgs branch for the A theory: for genus g theory, there are $g(N-1)$ free U(1) vector hypermultiplets in the Higgs branch. So the dimension of the "Higgs" branch is equal to the number of hypermultiplets minus the vector multiplets, and then plus $g(N-1)$. Let's consider genus one Riemann surface with just one simple puncture, the graph and the mirror is shown in Figure 47a). This is just 4d $\mathcal{N}=2^*$ SU(N) theory, one should be a little bit careful here, the adjoint matter has dimension N^2. One can check that the Coulomb branch and Higgs branch dimensions of A and B matches by counting the dimension of "Higgs" branch of A by including the free U(1) vector multiplets.

There are three types of internal legs for the higher genus theory in the dual graph. For the first one, when we cut it, the number of loops of the graph is not changed. When we cut the second type of internal legs, the graph becomes two disconnected parts with loops; when we cut the third type of internal legs, there is only one part left and its number of loops is reduced by 1. When we add more fundamentals to the first two types of internal legs, the procedure of finding the mirror is the same as the genus zero case. In particular, for the second type of internal legs, the gauge group must be SU(N), adding a fundamental just introduces another U(N) group and a bi-fundamental.

For the third type of internal legs, there is a subtle point when we add just one fundamentals. Consider the example in Figure 47, we add a "D5" brane on the internal leg as depicted in Figure 47b). Do S-dual on the graph, the "D5" brane becomes a "NS5" brane, which cuts the original U(N) group into two U(N) groups, however, these two U(N) groups are connected by a single junction which must be a single U(N) since in the mirror only the diagonal part of the junction is survived. The mirror has just one U(N) node. We need one modification: we replace the adjoint of SU(N) with adjoint of U(N), and we let the central node be U(N) instead of SU(N).

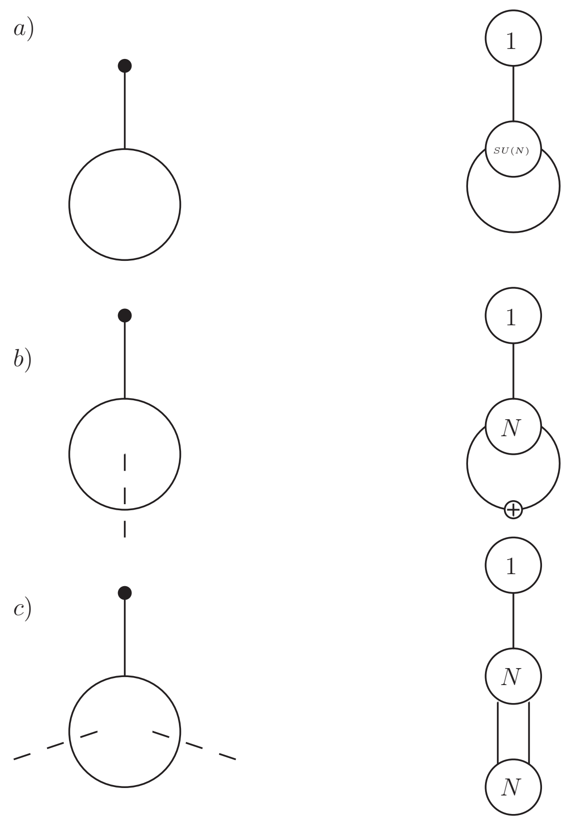


Fig. 47. a): The graph representation of $\mathcal{N} = 2$ SU(N) with an adjoint is depicted on the left, the mirror theory is depicted on the right, the loop attached on SU(N) is the adjoint of SU(N). b): On the left, we add one more fundamental to theory a); There is a cross on the adjoint which means that here is the adjoint on U(N), also the central node is U(N) group. c): We add two fundamentals to theory a); The mirror is depicted on the right, no U(1) is projected out.

See Figure 47b) for the mirrors.

Let's compare the Coulomb branch and Higgs branch dimension of theory $\tilde{A}$ and $\tilde{B}$. Now the Higgs branch of $\tilde{A}$ is just the difference between the hypermultiplets and vector multiplets. It is easy to see the dimensions matches. There are two mass parameters for the theory $\tilde{A}$ and in the mirror there are two U(1) factors so we have two FI parameters as it should be.

When we add two fundamentals, there is no adjoint in the mirror, we have two U(N) gauge groups, see Figure 47c). In general, when we add k fundamentals, there are k U(N) gauge groups in the mirror.

In general, for genus g theory, when we add just one fundamental on one of the handle, in the mirror, the central node is changed to U(N) and one of the adjoint of SU(N) is changed to the adjoint of U(N); After doing that, there is only $(g-1)(N-1)$ free U(1)s in the "Higgs" phase of $\tilde{A}$. If we add another fundamental to a different handle, we simply change one of the adjoint of SU(N) to U(N), the number of free U(1) is reduced by $(N-1)$. However, when we add one fundamental on each handle, there is not enough FI terms in the mirror, what happens we believe is that there is hidden "FI" terms which appear only in the IR. There are a total of $(g-1)$ hidden "FI" terms.

We can add arbitrary number of fundamentals to any of the weakly coupled gauge group. A genus two example is shown in Figure 48.

b. D_N theory

The above analysis can be extended to D_N theory. Let's first discuss the definition of the "good", "bad", "ugly" for the USp and SO gauge theory. For SO(k) gauge a)

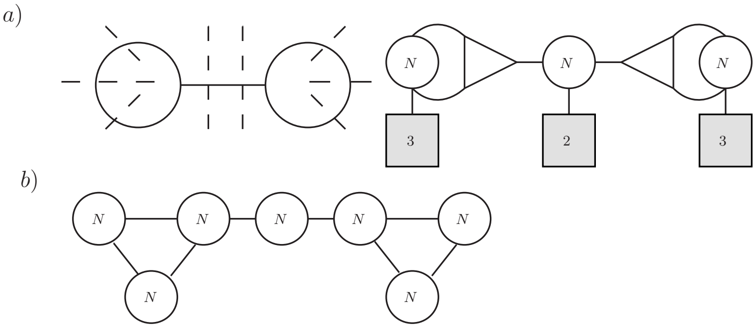


Fig. 48. Top: A genus two theory with several fundamentals, the generalized quiver representation is shown on the left . Bottom: Mirror theory of (a).

theory with n_f flavors, we define the excess number

$$e = n_f - k + 1. \tag{6.44}$$

The theory with $e \geq 0$ is called "good". For USp($2t$) theory with n_f flavors, the excess number is defined as

$$e = n_f - 2t - 1. \tag{6.45}$$

The theory with $e \geq 0$ is "good". The above theories are called "balanced" if $e = 0$, notice that the balance condition for three dimensional theory is different from the conformal invariant condition for the four dimensional theory. So the conventional conformal orthosymplectic quiver is not a "good" quiver, which is different from the unitary case. In fact, the SO nodes are "bad".

Similarly, a quiver with alternative USp and SO nodes are called "good" quiver if $e_i \geq 0$ for every node in the quiver; It is called "balanced" if $e_i = 0$ for every node. The global symmetry of Coulomb branch is enhanced by monopole operators for a chain of balanced orthosymplectic quiver with P nodes, the global symmetry is in general enhanced to $SO(P+1)$. However, if the first node on the chain is $SO(2)$, the

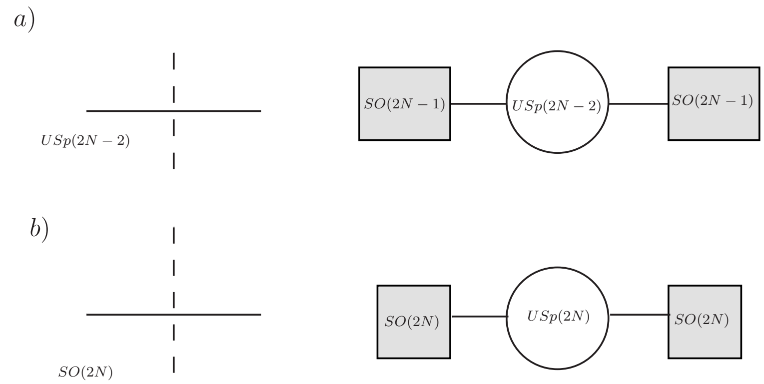


Fig. 49. Top: The addition of a "D5" brane to a USp leg, its mirror is depicted on the right. Bottom: The addition of a "D5" brane to a SO leg and its mirror.

global symmetry on the Coulomb branch of a chain with P nodes is $SO(P+2)$.

For the theory considered in [25], there is USp global symmetry for the A theory, but there is no balanced orthosymplectic quiver with enhanced USp flavor symmetry, so in general the mirror quiver is a "bad" quiver.

There are two types of internal legs for the theories considered in [25]. We add some full "D5" branes to the internal leg and the mirror of one "D5" brane is shown in [61] using brane splitting, we reproduce it in Figure 49.

To find the mirror theory $\tilde{B}$, we can not simply extend our analysis for the unitary group case, since the mirror B is already a bad quiver. The theory $\tilde{B}$ should have the same Higgs branch dimension as the theory $\tilde{B}$. The graph mirror has the same Higgs branch as $\tilde{B}$. Now we would like to do some manipulation on the "bad" node so that the Higgs branch dimension is not changed. Let's consider a USp($2k$) node, the Higgs branch contribution (include all the matter attached on it) is

$$N_f 2k - (2k^2 + k) = \frac{1}{2} 2k(2N_f - 2k - 1). \tag{6.46}$$

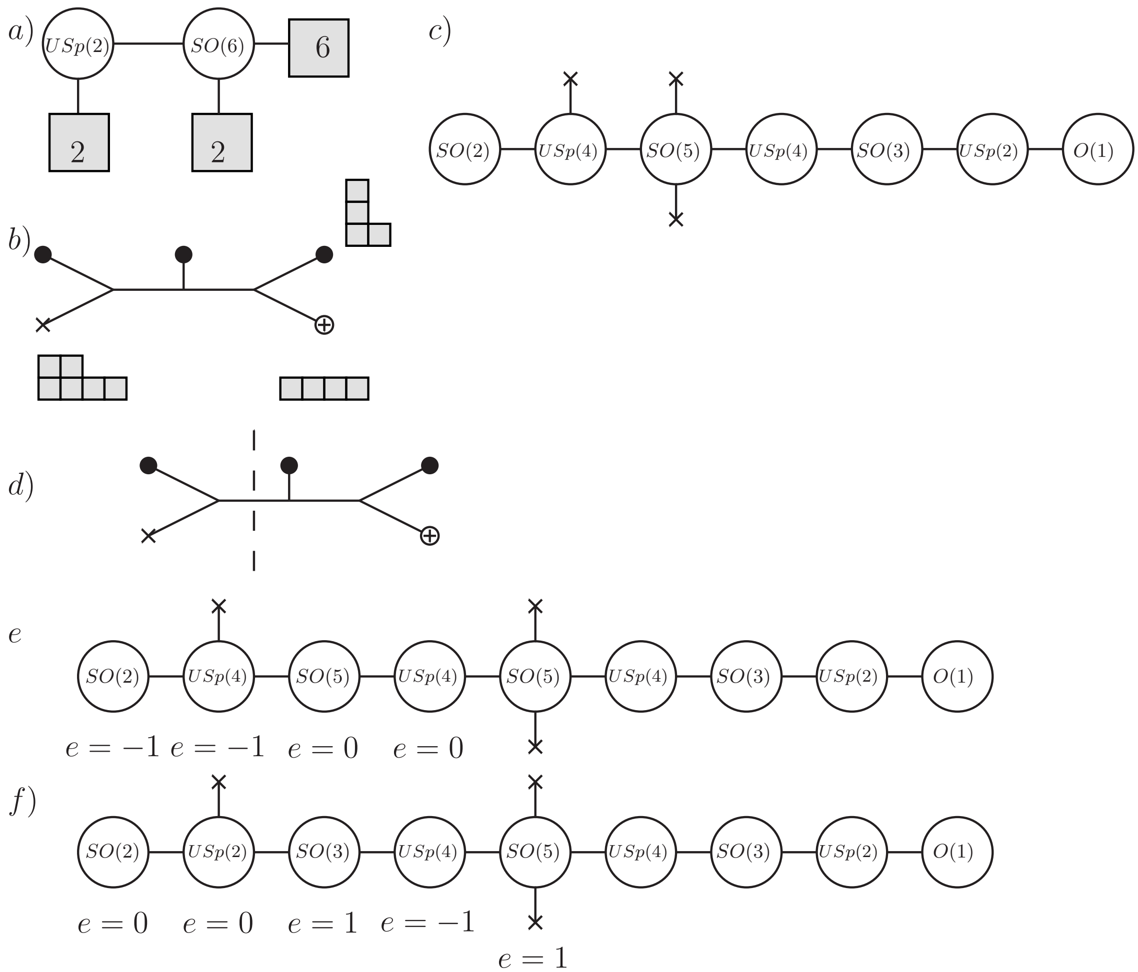


Fig. 50. a); An orthosymplectic quiver which is conformal in four dimension. b): The graph representation of the three dimensional theory; We indicate the Young Tableaux here. c): The mirror theory B. d): We add one full "D5" brane on the internal leg on USp(2) leg. e): The naive mirror of the (d), we indicate the excess number for some of the relevant nodes; We can do the rank manipulation on USp(4) node. f): We change the USp(4) node to USp(2), and then change the adjacent SO(5) node to SO(3), the new excess number is indicated. This is the mirror theory $\tilde{B}$.

For a bad quiver, we may wonder to replace its rank $2k$ with $2N_f - 2k - 1$ so the contribution to the Higgs branch is the same. We can not do this since $2N_f - 2k - 1$ is a odd number. We can try to replace $2k$ with

$$2k^{'} = 2N_f - 2k - 2 = 2k + 2(N_f - 2k - 1) = 2k + 2e, \tag{6.47}$$

This is similar to the unitary case. However, after doing this, the Higgs branch contribution of this node is increased by 2e.

For the $SO(N_c)$ gauge theory, one can do the similar calculation as above, the final result is that if we replace the rank of the gauge group by $N_c^{'} = N_c + 2e$. The Higgs branch contribution of this node is increased by $-2e$.

Suppose we have a node with negative excess number e, after doing the manipulation, the Higgs branch dimension is changed by $2e$ (we assume that this node is USp type, SO type is similar), assume one of the adjacent node originally has excess number e_1, its excess number becomes $e_1^{'} = e_1 + e$. First, we should ensure that $e_1^{'} < 0$, and we replace its rank follows our rule, the Higgs branch dimension is changed by $-2e_1^{'}$, to cancel the change of the USp node, we have the relation

$$e_1 = 0. \tag{6.48}$$

One may wonder we can do the manipulation on both adjacent nodes to cancel the contribution, a little bit calculation shows that this is possible only in the case $e_1 = e_2 = 0$.

Our conjecture is that we only do the manipulation on those “bad” nodes one of whose adjacent node is “balanced”. We continue this process until no nodes satisfy this condition. This constraint makes sense, the “bad” node on the original quiver leg does not satisfy this condition so we do not need to do the manipulation.

Let’s also give a simple example to illustrate the idea. We take an theory A for which we have a lagrangian description (the general case are really the same). See Figure 50 for the details. The flavor symmetry on the Coulomb branch of the quiver in Figure 50f) is changed from SO(2) to SO(4), which is exactly the flavor symmetry on USp(2) node in Figure 50a) (original, we have two half-hypermultiplets on USp(2) node, the flavor symmetry is SO(2), after adding two more half-hypermultiplets, the flavor symmetry is changed USp(4)).

With this construction, We can reproduce the results from [69]. In fact, we have constructed a large class of new mirror pairs for which the theory $\tilde{A}$ involves strongly coupled matter.

An important application of this "D5" brane probe is that we can read the weakly coupled gauge group by counting the change of the dimension of the Coulomb branch of the mirror. One subtly we should mention is that for the USp leg, we just add one "D5" brane and count the change of the Coulomb branch dimension in the mirror. However, the SO node is "bad", so we need to first add one "D5" brane to make it good, and then add another "D5" brane to probe the rank of the gauge group. This is the only tool we know to completely determine the generalized quiver from D_N theory. One may also determines the matter contents as we do for the unitary case.

One can also extend those consideration to the higher genus theory of the D_N type.

3. Gauging U(1)

For all the theories considered in [25] and this paper, the theory A has SU(k) gauge groups while in the mirror B, there is no fundamentals attached on any quiver node. In this subsection, we will show how the mirror changes if some of the U(1) symmetry of theory A is gauged.

The rule is quite simple, when there is a U(1) symmetry in the A theory, there is a U(1) gauge group in the mirror. If we gauge the U(1) symmetry of A to get theory $\tilde{A}$, the Higgs branch dimension of $\tilde{A}$ is decreased by 1 while the Coulomb branch dimension is increased by 1 comparing with A. To match this counting, we should ungauge the U(1) node of B to get a theory $\tilde{B}$ whose Higgs branch is increased by 1 while the Coulomb branch is decreased by 1 comparing with B . This is in agreement with the prediction of mirror symmetry.

The theory $\tilde{A}$ loses a mass parameter while $\tilde{B}$ loses a FI parameter; The Higgs branch of $\tilde{A}$ loses a U(1) global symmetry while $\tilde{B}$ loses a U(1) global symmetry in Coulomb branch. Those are also in agreement with Mirror symmetry.

Consider the example in Figure 47c), when we gauge the U(1) symmetry, the A theory is U(N) theory an adjoint and two fundamentals, the B theory is the quiver in the right of Figure 47c) with the U(1) node uncaged, this is in agreement with the result in [64].

With this gauging trick, we can add some fundamentals on the central nodes (nodes with at least three instrumentals attached on it)in the mirror. Notice that we can not add fundamentals on the nodes on quiver tail attached to the central node.

4. General quiver tail

In the above generalizations, we do not change the boundary condition on the external leg of the graph, so the quiver tail is the same. Our theory A is simply a chain of simple unitary nodes coupled with fundamentals (sometimes antisymmetric matter) and strongly coupled matter. Theory A does not have a lagrangian description in general. Our theory B is always a standard quiver gauge theory.

In this subsection, we want to change the boundary condition so that we have general quiver tail attached to the central node. In particular, we will allow quiver tails whose nodes can have the fundamental hypermultiplets.

To proceed, we need to reinterpret the graph representation of the theory A in terms of $\mathcal{N} = 4$ SYM on a half space. The general boundary condition of $\mathcal{N} = 4$ theory can be labeled by (ρ, H, B) [68] where ρ is the homomorphism $\rho : su(2) \to g$, H is the commutate of the ρ in G, B is a three dimensional theory which is coupled to H.

Consider a theory A whose graph representation only has one internal leg and four external legs, and we assume that the weakly coupled gauge group on the internal leg is SU(N). This theory can be interpreted as gauging two matter systems T_1, T_2 which are represented by trivalent graphs . As we proved in [23], these two matter systems must have SU(N) flavor symmetry, the gauge group is derived by gauging the diagonal SU(N) symmetry. These two matter systems are irreducible (they have Coulomb branch parameters with dimension N), so their 3d mirrors are "good" quivers. Consider one end of the internal leg, the boundary condition is just $(0, \mathrm{SU}(N), T_1)$, similar thing can be said at the other end.

The mirror theory B can be found by first finding the mirror of the matter system T_1, T_2 and then gluing them together. The mirror of the matter system is also the star-shaped quiver with a quiver tail which is $T(SU(N))$ (see [61] for its definition) corresponding to the SU(N) flavor symmetry. The result of gluing is just annihilate those two $T(SU(N))$ tails and we are left with only one central node, we refer this as the gluing process to get the mirror B, which is a counterpart of gauging for the original theory A. See Figure 8 in [25].

Let's count the Coulomb branch and Higgs dimension of the theory from gluing T_1 and T_2. For theory A, the Coulomb branch dimension is the sum of the Coulomb branch of three parts $(C_1 + C_2 + (N-1))$, where C_1 and C_2 is the Coulomb branch dimension of T_1 and T_2. The Higgs branch dimension is the $(H_1 + H_2 - (N^2 - 1))$, here H_1 and H_2 is the Higgs branch dimension of T_1, T_2. In the mirror, before the

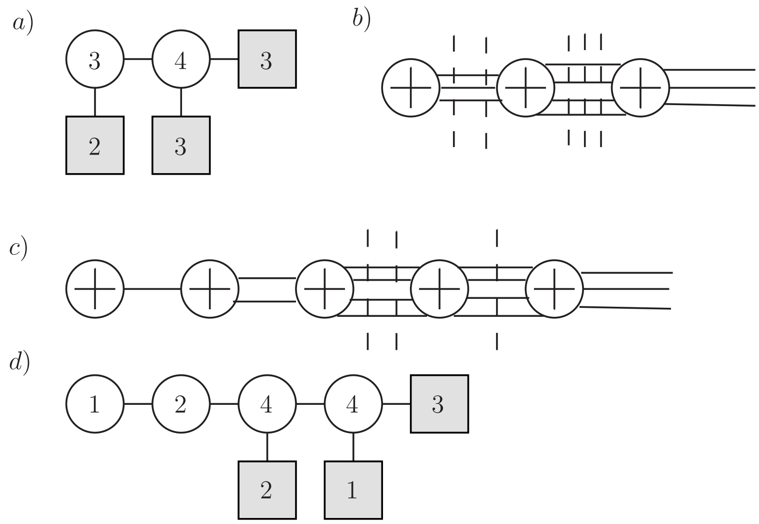


Fig. 51. a): A linear quiver which we called theory B in the main text, it has a SU(3) symmetry which is used to coupled to the $T_(SU(N))$. b): The brane construction for the quiver in (a), here crosses represent NS5 branes, vertical dash lines represent the D5 brane, horizonal lines are D3 branes. c): The S-dual brane configuration of (b), we have done a brane rearrangement so there is a gauge theory interpretation. d): The quiver representation of (c), which is the theory $B^\vee$ in the main text.

annihilation, the Coulomb branch dimension is simply $H_1 + H_2$ by mirror symmetry. After annihilating two $T(SU(N))$ legs, the Coulomb branch dimension is decreased by $N^2 - 1$; and the Higgs branch dimension is increased by $N - 1$, this agrees with theory A using mirror symmetry.

Now we can replace the theory T_1 and T_2 by any other "good" theories $B^\vee$ with SU(N) flavor symmetry to form a theory $\tilde{A}$. As long as we know the mirror of $B^\vee$, we may find the mirror theory $\tilde{B}$. If the mirror of T has a quiver tail $T(SU(N))$, nothing will prevent us to find the mirror $\tilde{B}$ by simply annihilating the $T(SU(N))$. Interestingly, in [61], a large class of those theories has been found.

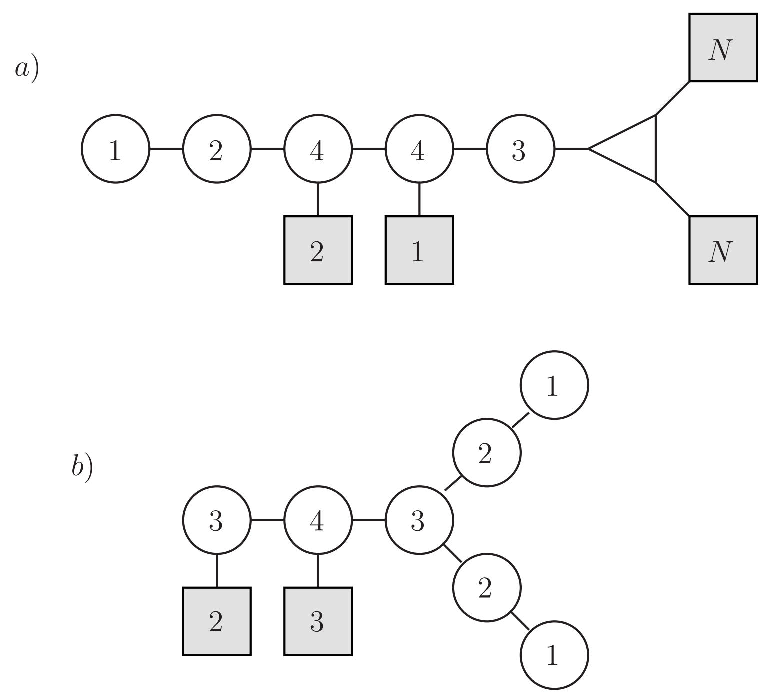


Fig. 52. Top: A quiver with T_3 factor on one end of SU(3) gauge group, on the other end, it is coupled to the theory $B^\vee$ as depicted in Figure 51d). Bottom: The mirror to theory (a), we use the quiver B in Figure 51a).

Let's give a short review of their results. Suppose we have a boundary condition $(0, \mathrm{SU}(N), B)$, assume the dual boundary condition has full gauge symmetry. the dual boundary condition is $(0, \mathrm{SU}(N), B^\vee)$ where $B^\vee$ is a SCFT living in the boundary. The mirror $\tilde{B}^\vee$ is

$$\tilde{B}^\vee = B_{SU(N)\times SU(N)} T(SU(N)). \tag{6.49}$$

We assume that this is a "good" quiver, so the $\mathrm{SU}(N)$ symmetry on the Coulomb branch of $T(SU(N))$ is the $\mathrm{SU}(N)$ symmetry of the Higgs branch of $B^\vee$.

Now we replace one of theory T_1 with theory $B^\vee$, the mirror is simply to replace two quiver tails from T_1 by B. The above consideration is quite general. Here we consider the case where the theory B is a linear quiver, since we require the quiver in 6.49 to be good quiver, the first node of this quiver should have rank $N_c \geq N+1$. B has a $NS5 - D5 - D3$ brane constructions, we can find theory $B^\vee$ by doing S-duality on the brane systems. To have a gauge theory interpretation, we may need to rearrange the branes, see [62, 61] for more details. See Figure 51 for an example.

Now let's consider a SU(3) theory coupled with two T_3 theory, we replace one of T_3 with the theory $B^\vee$, which we call the theory $\tilde{A}$, the mirror theory $\tilde{B}$ can be found from our general recipe. See Figure 52 for an example.

The theory T_N can play an interesting role, for each $\mathrm{SU}(N)$ flavor symmetry, we can attach a general quiver tail. then the mirror is a star-shaped quiver with three general quiver tails. In fact, using T_N theory, we can construct the mirror theory with any number of general quiver tails.

The above consideration can be extended to the case even if the quiver is "bad", the mirror is still given by a star-shaped quiver as we described earlier, but the gauge group on the leg is not $\mathrm{SU}(N)$ but broken down to a subgroup, we can use the method in this subsection to determine the theory A.

5. Irregular singularity

The mirror theory we considered so far are almost linear quiver with only one bi-fundamental connecting the quiver nodes. The shape of the quiver is quite simple. In this subsection, we will see more general type of quivers.

There are other four dimensional $\mathcal{N} = 2$ field theories constructed from six dimensional $(0, 2)$ SCFT. In the examples discussed in [25], one consider the compactification on Riemann surface with regular singularities, this defines a four dimensional $\mathcal{N} = 2$ SCFT [23]. However, one can also consider irregular singularities [24, 56] on the Riemann surface. This defines a four dimensional theory A which we will study in detail elsewhere. In this subsection, we will study the mirror theory for some of those theories which have already been mentioned in [24].

The moduli space of the Hitchin equation with irregular singularity is the Coulomb branch of the four dimensional theory on a circle with radius R. In the deep IR limit, there is a three dimensional $\mathcal{N} = 4$ SCFT which we call theory A. We want to find the mirror theory B. In the case of regular singularities, we attach a quiver tail to each of the singularity and then glue the common $\mathrm{SU}(N)$ nodes together.

The procedure for the irregular singularity case is quite similar. We need to define a quiver to the irregular singularity. After doing this, we glue those quiver tails of the regular singularities to the quiver associated with the irregular singularity. So the problem is reduced to find the quiver tail of the irregular singularity. The general story of irregular singularity is quite complicated and we do not intend to consider the general case in this paper. We only consider some simple cases and discuss the general theory elsewhere. Although as simple as the theory A we consider in this paper, the mirror seems to have some new features: more than one bi-fundamentals and exotic quiver shape. We should mention that, in mathematics literature, the moduli space of Hitchin's equation in complex structure J has been given a quiver approximation [58], in physics language, we extend this observation to the level of Hyperkahler structure, moreover, the Coulomb branch of the quiver is identified with the Higgs branch of the theory A, which is not recognized in mathematics literature.

a. A_1 theory

The SU(2) Hitchin system defined on Riemann surface (in this paper, we only consider Riemann sphere) has only two types of irregular singularities [56], we write the form of the holomorphic Higgs field

$$\Phi = \frac{A}{z^n} + ...$$
$$\Phi = \frac{A}{z^{n-1/2}} + ..., \tag{6.50}$$

where A is a diagonal matrix. we call them Type I and Type II singularity respectively. When we put such a singularity on the Riemann surface, we get a four dimensional $\mathcal{N} = 2$ theory A. One can add other regular singularities on the Riemann sphere. The moduli space of Hitchin's equation in complex structure I has the famous Hitchin's fibration, which is identified with the Seiberg-Witten fibration of the four dimensional theory. In particular, the Coulomb branch dimension of four dimensional theory has half the dimension of the Hitchin's moduli space. The spectral curve of the Hitchin system is

$$det(x - \Phi) = 0, \tag{6.51}$$

which is just the Seiberg-Witten curve. The total dimension of Hitchin's moduli space is equal to the contribution of each singularity and minus the global contribution $2(dimG - r) = 6$, here $G = \mathrm{SU}(2)$, r is the rank of the gauge group. The local contribution of regular singularity is 2 [56], and $2n$ for the irregular singularity. We only consider just one irregular singularity on this paper(when there are more than one irregular singularities, the mirror is not a quiver).

The mass parameter for the four dimensional theory is encoded in the residue of the Higgs field. So for type I singularity, there is one mass parameter. However, there is no mass parameter for type II singularity since the residue term is not allowed because of monodromy. We want to attach a quiver with these irregular singularities. In the case of regular singularity, the dimension of the Higgs branch of the quiver tail only accounts for the local dimension of the singularity. In the case of irregular singularity, we should include the global contribution to the irregular singularity: the Higgs branch of the quiver should have dimension $n-3$. The quiver should have one U(1) factor for Type I node and no U(1) factor for type II node, since the FI term for the U(1) would correspond to the mass parameter (these are not true in general, since we have exotic IR behavior for some theories, in these cases, one can not tell what exactly happens in the IR from the UV theory).

With these considerations, we have the following conjecture for the quiver attached on the irregular singularity: for the Type I singularity, we have two nodes with U(1) group, and there are $(n-2)$ bifundamentals connecting them, one of the U(1) is decoupled, so we only have one FI term; There are $n-3$ mass terms for the bi-fundamentals, this means that the original theory has $n-3$ "hidden" FI terms. The origin of these "hidden" FI terms will be discussed elsewhere. One example is shown in Figure 53a).

For Type II singularity, if $n \geq 5$ we have only one U(1) node with $n-2$ lines connect to itself, these are the adjoints for the U(1) which are just the fundamentals. This is exactly like the mirror for the high genus theory in the context of regular singularities [25]. There are also enhanced global symmetry in the mirror and there are extra $n-3$ mass parameters which correspond to the "hidden" FI term in the original

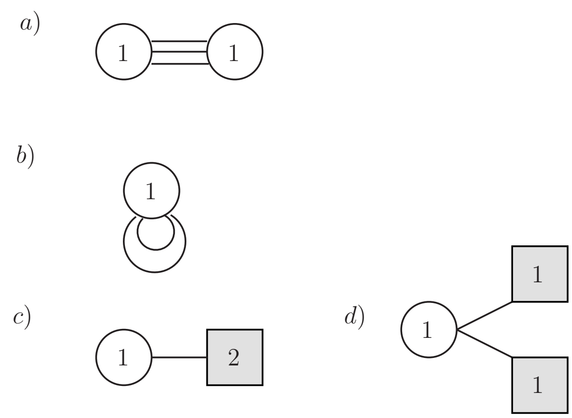


Fig. 53. a): The quiver for type I irregular singularity, here $n = 5$. b): The quiver for type II irregular singularity, here $n = 5$, here we have adjoint matter on U(1). c): The quiver tail for a regular singularity. d): We spray the SU(2) symmetry into two U(1)s.

theory A. Follow the analogy with the higher genus theory, $n = 4$ corresponds to genus 1 case. In the genus one case, the massless limit has enhanced supersymmetry, and the mirror has only one adjoints while the massive limit has an extra U(1) node. For the irregular singularity here, we conjecture that the mirror corresponds to the massless limit of the genus 1 case, so we only have one adjoint, the mirror is indeed U(1) with one fundamental, we will confirm this later. Similarly, for $n = 3$, there is only one U(1) node, in this case, there is no meaning to consider this irregular singularity alone, this is only a recipe to form the mirror quiver when there are extra regular singularities. For this class of singularities, we show one example in Figure 53b).

When there are other regular singularities, the quiver tail is shown in Figure 53c). We spray the node as in Figure 53d). To connect the regular singularity to irregular singularity, we just gauge the U(1) node: In type I case, they are gauged separately;

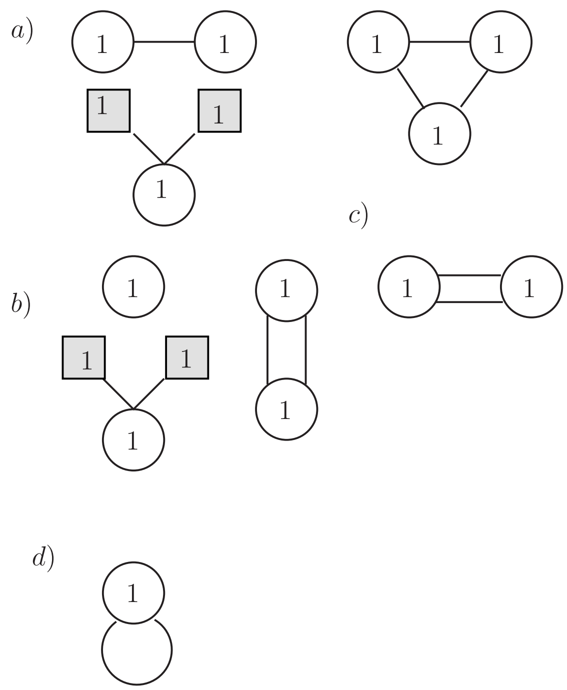


Fig. 54. a): The mirror for the A_2 Argyres-Douglas point, there is one Type I irregular singularity with $n = 3$ and one regular singularity, we glue them together to form a quiver on the right. b): For A_1 Argyres-Douglas point, there is a type II irregular singularity with $n = 3$ and a regular singularity, after gluing them, we get a A_1 affine dynkin diagram. c): Another representation of A_1 Argyres-Douglas point, only one Type I irregular singularity with $n = 4$ needed, the resulting quiver is the same as representation (b). d): For A_0 Argyres-Douglas point, only one Type II singularity with $n = 4$ is needed, the resulting mirror is just the A_0 affine dynkin diagram.

in type II case, they are gauged on the same node. Since the Higgs branch of this quiver tail accounts the local dimension of the regular singularity to the Hitchin's moduli space even after spraying, the Higgs branch of the whole quiver is the same as the Coulomb branch of the theory A.

We can consider some examples studied in [24]. In that paper, we claim that the Argyres-Douglas A_2 theory is derived with one Type I irregular singularity with $n = 3$ and a regular singularity. The Seiberg-Witten curve as from 6.51 is

$$x^2 = z^2 + u_1 z + u_2 + \frac{u_3}{z} + \frac{m^2}{z^2}. \tag{6.52}$$

We put the irregular singularity at $z = \infty$ and regular singularity at $z = 0$. Let's count the scaling dimension of the operators in the curve. The Seiberg-Witten differential is $\lambda = xdz$; We require its dimension to be 1, together with the form of the Seiberg-Witten curve, we have the scaling dimension of x and z:

$$[x] = \frac{1}{2}, [z] = \frac{1}{2}. \tag{6.53}$$

One can easily get the scaling dimension of the operators: $[u_1] = \frac{1}{2}, [u_2] = 1, [u_3] = \frac{3}{2}, [m] = 1$, which is the same as the A_2 Argyres-Douglas (AD) points as shown in [55]. The Coulomb branch dimension of this theory is 1, and the Higgs branch dimension is 2. The flavor symmetry is SU(3) which is not easy to see from our six dimensional description, since there is only manifest SU(2) $\times$ U(1) flavor symmetry. We will see below that the we can see the enhanced symmetry from the mirror.

The mirror quiver is depicted in Figure 54a), we show how to glue the quiver from the regular singularity and the irregular singularity together. Let's check that it gives the correct mirror description: The Higgs branch dimension is 1 and Coulomb branch is 2, which agrees with the prediction from mirror symmetry. Since the mirror quiver has a chain of balanced quiver with two nodes (one U(1) is decoupled), the symmetry in the Coulomb branch is SU(3) which is exactly the flavor symmetry of the A_2 theory.

For A_1 AD points, there are two construction: one Type II irregular singularity with $n = 3$ and a regular singularity; or we can have just one Type I irregular singularity with $n = 4$. For the first representation, the spectral curve is

$$x^2 = z + u_1 + \frac{u_2}{z} + \frac{m^2}{z^2}. \tag{6.54}$$

We put irregular singularity at $z = \infty$ and regular singularity at $z = 0$. The scaling dimension is $[x] = \frac{1}{3}, [z] = \frac{2}{3}$. The scaling dimension of the spectrum is $[u_1] = \frac{2}{3}, [u_2] = \frac{4}{3}, [m] = 1$, which is the same as the A_1 points [7].

For another representation, the spectral curve is

$$x^2 = z^4 + u_1 z^2 + m z^3 + u_2. \tag{6.55}$$

We have shift the origin so z term is absent. The spectrum is $[u_1] = \frac{2}{3}, [m] = 1, [u_2] = \frac{4}{3}$, which is the same as the above representation.

The A_1 AD point has Coulomb branch dimension 1 and Higgs branch dimension 1, the flavor symmetry is SU(2). The 3d mirror B are the same as we can see in Figure 54b), this also justifies that these two descriptions are equivalent. The mirror has Coulomb branch dimension 1 and Higgs branch dimension 1, the symmetry on the Coulomb branch is SU(2), these are all in agreement with the mirror symmetry.

A_0 theory is defined on a sphere with a Type II irregular singularity with $n = 4$. The spectral curve is

$$x^2 = z^3 + u_1 z + u_2. \tag{6.56}$$

The spectrum is $[u_1] = \frac{4}{5}, [u_2] = \frac{6}{5}$, which is the same as shown in [7]. The Coulomb branch of A_0 theory is 1 and the Higgs branch is 0.

The mirror for A_0 theory is shown in Figure 54c), which is just U(1) with one fundamentals. In the deep IR, it is just a twisted free hypermultiplet, so the Coulomb branch is 1, and the Higgs branch is 0 (see e.g. [61]). In this case, the mirror symmetry does not match the Coulomb branch to Higgs branch, but match the Coulomb branch to Coulomb branch, this kind of phenomenon has also been observed in [75].

Notice that the mirror are just the affine dynkin diagram of the corresponding type for the Argyres-Douglas points. The singular fibre is classified by Kodaira with type $A_0, A_1, A_2, D_4, E_6, E_7, E_8$. Four dimensional superconformal field theory with these curves are found [54, 7, 48, 76]. The mirror theory of IR limit of the three dimensional cousin are just given by the corresponding ADE affine dynkin diagram. Interestingly, the Coulomb branch of these isolated SCFT in three dimensions are the ALE space of the corresponding type.

b. A_{N-1} theory

The classification of the irregular singularity for rank N theory is quite complicated. Here we only consider the most simple irregular singularity.

$$\Phi = \frac{A}{z^n} + ... \tag{6.57}$$

where A is the diagonal matrix with distinct eigenvalues. The Hitchin equation with this kind of singularity has been considered in detail in the gauge theory approach to Geometric Langlands program [56].

The local dimension of just one singularity is

$$n(dimG - r), \tag{6.58}$$

where r is the rank of the gauge group and G is SU(N) in the present context. The total dimension of the Hitchin's moduli space with just one such irregular singularity

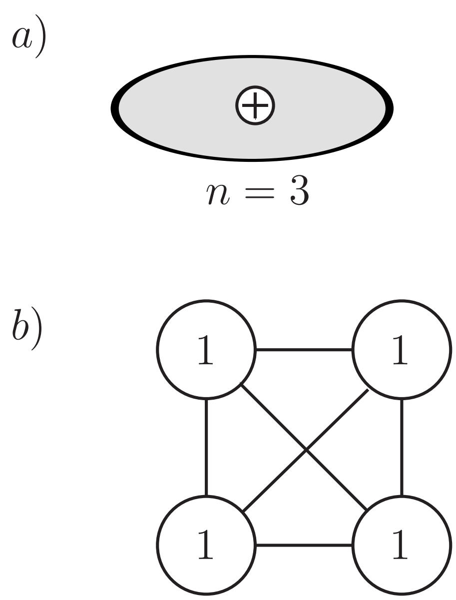


Fig. 55. Top: A four dimensional superconformal field theory derived from six dimensional $(0, 2)$ theory on a sphere with one irregular singularity. Bottom: The mirror to theory (a).

is

$$n(N^2 - N) - 2(N^2 - 1). \tag{6.59}$$

So the four dimensional theory A has Coulomb branch dimension

$$d_C = \frac{1}{2}n(N^2 - N) - (N^2 - 1) = \frac{1}{2}(n-2)(N^2 - N) - (N - 1). \tag{6.60}$$

There are a total of $N - 1$ mass parameters.

c. Three dimensional mirror for A_N theory

When we compactify six dimensional theory on a sphere with such a singularity, we get a four dimensional SCFT A. The spectrum can be worked out similar as the AD points, we will give it elsewhere, here we consider its mirror B. The mirror to this theory is quite simple, there are N nodes with U(1) gauge groups and there are $(n-2)$ bi-fundamentals connecting each pair of nodes. The Higgs branch of this quiver B is

$$(n-2)\frac{1}{2}(N^2 - N) - (N - 1), \tag{6.61}$$

which is exactly the Coulomb branch dimension of A. See Figure 55 for an example.

CHAPTER VII

4D-2D CORRESPONDENCE*

The four dimensional theory is completely determined by the data defined on the two dimensional Riemann surface. It is found by AGT [19] that for SU(2) case, the Nekrasov partition function of four dimensional gauge theory is identified with the conformal block of the Liouville theory. This rather striking relation deserves further investigation.

S duality plays a central role in understanding four dimensional gauge theory. While crossing symmetry and modular invariance is central in studying two dimensional conformal field theory. I am going to study further the relation between these two fundamental concepts in 4d and 2d theory.

A. Crossing symmetry and modular invariance in conformal field theory

Let's first review how the dual string model was proposed [30]. Consider an elastic scattering amplitude with incoming spinless particles of momentum p_1, p_2 and outgoing spinless particles of momentum of p_3, p_4 (see Figure 56). The conventional Mandelstam variables are

$$s = -(p_1 + p_2)^2, \quad t = -(p_2 + p_3)^2, \quad u = -(p_1 + p_4)^2. \tag{7.1}$$

Consider first the t channel contribution. There are various particles with mass

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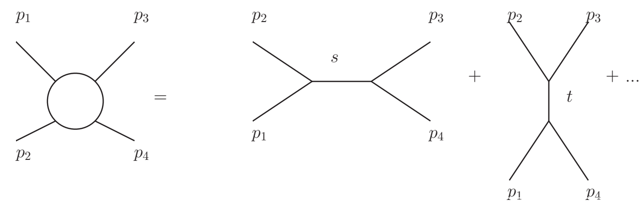

Fig. 56. An elastic scattering with incoming particles of momentum p_1, p_2 and outgoing particles of momentum p_3, p_4. We indicate the contribution from s channel and t channel. The field theory amplitude is constructed from the sum of those contributions.

M_J and spin J which might be exchanged:

$$A(s,t) = -\sum_J \frac{g_J^2(-s)^J}{t - M_J^2}. \tag{7.2}$$

We have the following similar amplitude if we consider s channel:

$$A^{'}(s,t) = -\sum_J \frac{g_J^2(-s)^J}{s - M_J^2}. \tag{7.3}$$

Two remarkable properties of the scattering amplitude are that the above sums are infinite and these two amplitudes are equal to each other $A(s,t) = A^{'}(s,t)$. The last property which is called $s-t$ duality motivates the proposed Veneziano amplitude.

It is well known that the Veneziano amplitude can be derived from two dimensional (world sheet) string theory. The infinite sum is due to the infinite number of states in mass spectrum. The $s-t$ duality is simply the crossing symmetry of the four point function of conformal field theory. This crossing symmetry is also equivalent to the associativity of the OPE on the world sheet:

$$A_i(\zeta)A_j(0) = \sum_k C_{ij}^k(\zeta)A_k(0). \tag{7.4}$$

We briefly summarize some properties of CFT. The Virasoro algebra is

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{7.5}$$

here c is the central charge and L_m are generators of conformal symmetry. The representations of this algebra are labeled by primary states which satisfy:

$$L_0|V_\alpha >= \Delta_\alpha|V_\alpha >, \quad L_n|V_\alpha >= 0, \;\; n > 0, \tag{7.6}$$

Δ_α is the conformal dimension of this primary state. The other states of this representation are represented as:

$$L_{-k_n}L_{-k_{n-1}}....L_{-k_1}|V_\alpha >, \tag{7.7}$$

here $k_n \geq k_{n-1}... \geq k_1$. These secondary states have conformal weights $\Delta = \Delta_\alpha + |Y|$; here $|Y|$ is the total boxes of the Young Tableaux with rows $k_1, ...k_n$. The correlation functions involving the energy momentum tensor and secondary states are expressed by the correlation functions of the primary states.

The OPE of two primary states are given as [50]:

$$\begin{aligned}\phi_m(z,\bar{z})\phi_n(0,0) = \sum_p c^p_{nm} z^{\Delta_p - \Delta_n - \Delta_m} \\ \bar{z}^{\bar{\Delta}_p - \bar{\Delta}_n - \bar{\Delta}_m}\psi_p(z,\bar{z}|0,0).\end{aligned} \tag{7.8}$$

The most important dynamical information are c^p_{nm} and the conformal dimensions.

The four point function has the form

$$G^{lk}_{nm}(x,\bar{x}) =< k|\phi_l(1,1)\phi_n(x,\bar{x})|m >, \tag{7.9}$$

here we fixed the positions of three vertex operators as $0, 1, \infty$ and x is the projective

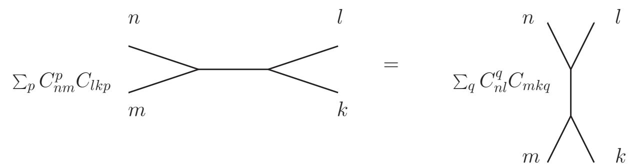

Fig. 57. Crossing symmetry of four point function in conformal field theory.

invariant variable. The crossing symmetry is

$$G^{kl}_{nm}(x,\bar{x}) = G^{mk}_{nl}(1-x,1-\bar{x}) = x^{-2\Delta_n}\bar{x}^{-2\bar{\Delta}_n}G^{lm}_{nk}(\frac{1}{x},\frac{1}{\bar{x}}) \tag{7.10}$$

Using OPE of $\phi_n\phi_m$, the four point function is

$$G^{lk}_{mn}(x,\bar{x}) = \sum_p c^p_{nm}c_{lkp}F^{lk}_{nm}(p|x)\bar{F}^{lk}_{nm}(p,\bar{x}), \tag{7.11}$$

here F is the conformal block which is entirely determined by the conformal symmetry(we give the s channel contribution). The crossing symmetry relates the contributions of different channels (see for instance Figure 57):

Next let's consider the one-loop partition function defined on a torus:

$$Z(\tau) = \sum_i q^{h_i - c/24}\bar{q}^{\tilde{h}_i - c}(-1)^{F_i}, \tag{7.12}$$

here i runs over all the states in CFT and F_i is the fermion number, $q = \exp(2\pi i\tau)$. It is well known that this partition function is needed to be invariant under the $SL(2,Z)$ modular group transformation of the torus. The high energy density states of this theory is determined by the central charge for compact unitary conformal field theory. For Liouville theory, there is a modification to this result, see [77].

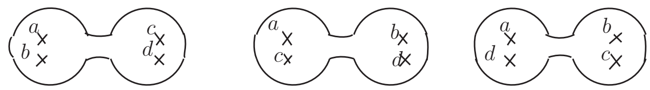

Fig. 58. Different degeneration limits of the punctured sphere, this corresponds to different weakly coupled S-dual theory of $\mathcal{N} = 2$ $SU(2)$ gauge theory with four fundamentals.

B. Superconformal field theories in four dimensions

It is proposed in [78] that a strongly coupled gauge theory can be described by a weakly coupled theory in which the elementary particles are the monoples of the original theory. This proposal is naturally realized in $\mathcal{N} = 4$ supersymmetric gauge theory and is extended to a $\mathrm{SL}(2, Z)$ duality group. Later, $D = 4$ $\mathcal{N} = 2$ SU(2) gauge theory with four fundamental hypermultiplets has also been shown to have the $\mathrm{SL}(2, Z)$ duality [1]. Gaiotto found an extremely useful way to describe the gauge structure of different S-dual frames of this theory. We use the $\mathrm{SU}(2)_a \bigotimes \mathrm{SU}(2)_b \bigotimes \mathrm{SU}(2)_c \bigotimes \mathrm{SU}(2)_d$ subgroup of full SO(8) flavor group. This theory can be realized as the six dimensional $(0, 2)$ A_1 theory compactified on a sphere with four punctures. The different S dual frames are realized as the different degeneration limits of this punctured sphere, see Figure 58. The Seiberg-Witten (SW) curve of this theory is written as

$$x^2 = \frac{u}{(z-1)(z-q)z}, \tag{7.13}$$

we have fixed the positions of three punctures and left an unfixed puncture which is identified with the gauge coupling constant. The SW curve for the mass deformed theory is $x^2 = \phi_2(z)$, where

$$\phi_2(z) = \frac{M_2(z)}{z^2(z-1)^2(z-q)^2} + \frac{U_2(z)}{z(z-1)(z-q)}. \tag{7.14}$$

For each puncture, we associate a mass parameter $m_{a,b,c,d}$ to it; the physical masses of the fundamentals are given by

$$m_{1,2} = m_a \pm m_b, \quad m_{3,4} = m_c \pm m_d. \tag{7.15}$$

The partition function of this theory on S^4 is given by [79]:

$$Z_{s^4} = \frac{1}{vol(G)} \int [da] e^{-\frac{4\pi^2 r^2}{g_{YM}^2}(a,a)} Z_{1loop} |Z_{inst}^N(r^{-1}, r^{-1}, a)|^2 \tag{7.16}$$

Here a is the parameter for the Coulomb branch and r is the radius of the sphere. The 1-loop part is given by:

$$Z_{1-loop}^{\mathcal{N}=2} = \frac{H(2a)H(-2a)}{\prod_{i=1}^4 H(a+m_i)H(a-m_i)}. \tag{7.17}$$

Here $H(x)$ is given by Barnes's G function $H(x) = G(1+x)G(1-x)$; m_i is the mass parameters for the four fundamentals and a is the Coulomb branch parameter. The instanton part of the partition function is identified with the Nekrasov instanton partition function [15] $Z_{inst}(\epsilon_1, \epsilon_2, a)$. Notice that for the S^4 case $\epsilon_1 = \epsilon_2 = \frac{1}{r}$.

Next, let's study the $\mathcal{N} = 4$ SU(2) theory. It is given by the six dimensional A_1 theory compactified on a smooth torus. The SL$(2, Z)$ duality of the gauge theory is interpreted as the SL$(2, Z)$ modular invariance of this torus. The full partition function of this theory is of the same form as formula (7.16). It is interesting to note that for $\mathcal{N} = 4$ U(M) gauge theory the one-loop part is trivial $Z_{1-loop} = 1$ and the full partition function for U(M) gauge theory is only from the instanton part (see the discussion in section 5 of [79], we change the normalization here though):

$$Z = C \frac{1}{|\eta(\tau)|^{2M}(2\pi\sqrt{\tau_2})^M}. \tag{7.18}$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1-q^k)$, $q = e^{2\pi i\tau}$.

C. S duality from the conformal field theory

AGT [19] made a conjecture that the full partition function of the above $\mathcal{N} = 2$ $SU(2)$ SCFT is equivalent to the correlation function of Liouville theory. It is shown by AGT that the instanton part of the gauge theory partition function is equal to the conformal block of the correlation function and the one-loop part and classical part correspond to the structure constant part of the correlation function. It is also argued that the energy momentum tensor of Liouville theory is related to the operator (7.14) (this can be seen from the classical uniformization problem with the punctured sphere).

The relation between the deformation parameters and the parameters in Liouville field theory is

$$\epsilon_1 = b, \quad \epsilon_2 = \frac{1}{b}, \tag{7.19}$$

here ϵ_1 and ϵ_2 are the deformation parameters in Nekrasov's instanton partition function. Notice that in order to use the partition function on S^4, we need to set $b = 1$.

We associate a exponential vertex operator $e^{\alpha_i \phi}$ to each puncture. We also associate a intermediate state $e^{\alpha\phi}$ to weakly coupled $SU(2)$ group with Coulomb parameter a. The exact relations between the parameters in gauge theory and Liouville theory are

$$\alpha_1 = m_a + \frac{Q}{2}, \alpha_2 = m_b, \alpha = a + \frac{Q}{2}, \alpha_3 = m_c + \frac{Q}{2}, \alpha_4 = m_d.$$

Here Q is the conventional parameter for the Liouville theory $Q = b + \frac{1}{b}$. See Figure 59 for the correspondence.

Now the crossing symmetry (7.10) of CFT states that the correlation function of different channels are related. When we consider the gauge theory, the different channels mean different S dual frames (see Figure 58). With the identification between

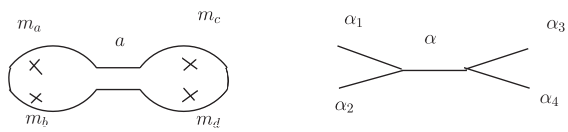

Fig. 59. Left: Riemann surface associated with the $SU(2)$ gauge theory in one particular S dual frame, the Coulomb branch parameter is a. Right: The s channel contribution to the four point function of Liouville theory.

the partition function of gauge theory and correlation function of CFT, we conclude that: The partition function of 4d SCFT in different S dual frames are related as in formula (7.10). Notice that the gauge coupling is identified with the position of the unfixed coordinate of the vertex operator, and the second identity in (7.10) relates theories with gauge couplings $q^{'} = \frac{1}{q}$. So this identity is a check of S-duality !

Next let's consider $\mathcal{N} = 4$ SU(2) theory, then we are tempting to identify the gauge theory partition function with the partition function of Liouville theory, the Liouville partition function can be calculated from (7.12):

$$Z(\tau) = V_\phi \frac{1}{2\pi\sqrt{\tau_2}|\eta(\tau)|^2}. \tag{7.20}$$

here V_ϕ is the zero mode contribution and is independent of τ, this is identified with the gauge theory partition function; we also need to identify $\alpha = 1 + a$ for intermediate state and the gauge coupling is $\frac{\pi}{g_{YM}^2} = \tau_2$ which is consistent with our previous identification. Comparing to (7.18) with $M = 2$, we can see that the $U(1)$ part contribution to gauge theory partition function is

$$Z^{U(1)} = C' \frac{1}{2\pi\sqrt{\tau_2}|\eta(\tau)|^2}. \tag{7.21}$$

CHAPTER VIII

CONCLUSION

This dissertation is only a first step towards a new understanding of quantum field theory in various dimensions. We want to point out several open problems which deserve further study.

Regarding the construction of four dimensional field theory, there are several remaining interesting problems. We are mainly studying field theory engineered from six dimensional A_N type theory, D_N theory is studied in some detail in [44], it is interesting to extend the same analysis to E_N case for which there is no simple type IIA brane construction. One can also consider orbifold Riemann surface with punctures, this might be useful in describing D_N type four dimensional quiver [80]. The most interesting question is to further study irregular singularity, only a very small set of singularities are used in this dissertation. It is possible to give a classification of four dimensional theory once the irregular singularity is fully understood.

Various extended objects of four dimensional theory deserves further study. The line operators for SU(2) generalized superconformal quiver gauge theory is classified and its property under S-dualtiy is also studied. This also uses the data on punctured Riemann surface [81]. It is interesting to extend to higher rank gauge theory. Surface operators [82] can be introduced and may shed light on the structure of gauge theory. Finally, the domain wall and the boundary condition needs further study, its S-dual property might be very interesting as what happens for $\mathcal{N} = 4$ super Yang-Mills theory.

Perhaps the most interesting problem is to find the wall crossing behavior of all these theories. The geometric setting of six dimensional construction is quite useful. Once again, the problem is reduced to the study of puncture Riemann surface. In fact, stable BPS particles are understood from the curve on the punctured Riemann surface [83]. Hopefully, the wall crossing phenomenon can also be extracted simply from the information on the puncture.

The gauge theory data on the Riemann surface is fully specified in this dissertation and the integrable system is defined for the gauge theory too. It is interesting to study in detail the AGT conjecture and NS conjecture. Maybe the crucial point is also the data on the Riemann surface, see the discussion in [84]. The AGT correspondence is extended to the case with insertion of line operators and surface operators on the gauge theory side [85, 86], it is definitely interesting to study the 2d-4d correspondence with the gauge theory constructed in this dissertation.

It is interesting to see if the same higher dimensional engineering is possible for four dimensional $\mathcal{N} = 1$ theory. It might also be possible study three dimensional Chern-Simons theory by turning on θ term of $\mathcal{N} = 4$ theory on the graph in our study of three dimensional mirror symmetry [87].

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